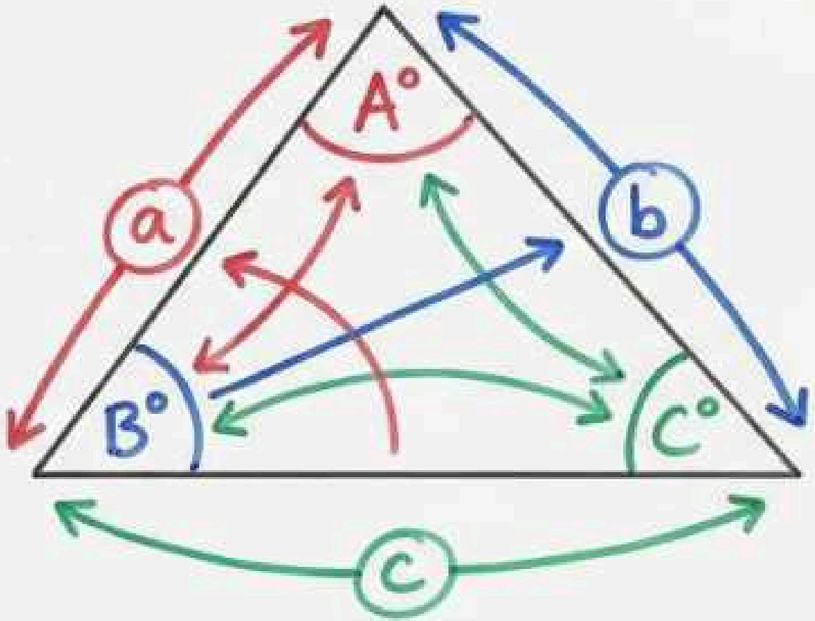
¶ 6.1

Solving Any Triangle: The Laws of Sines and Cosines

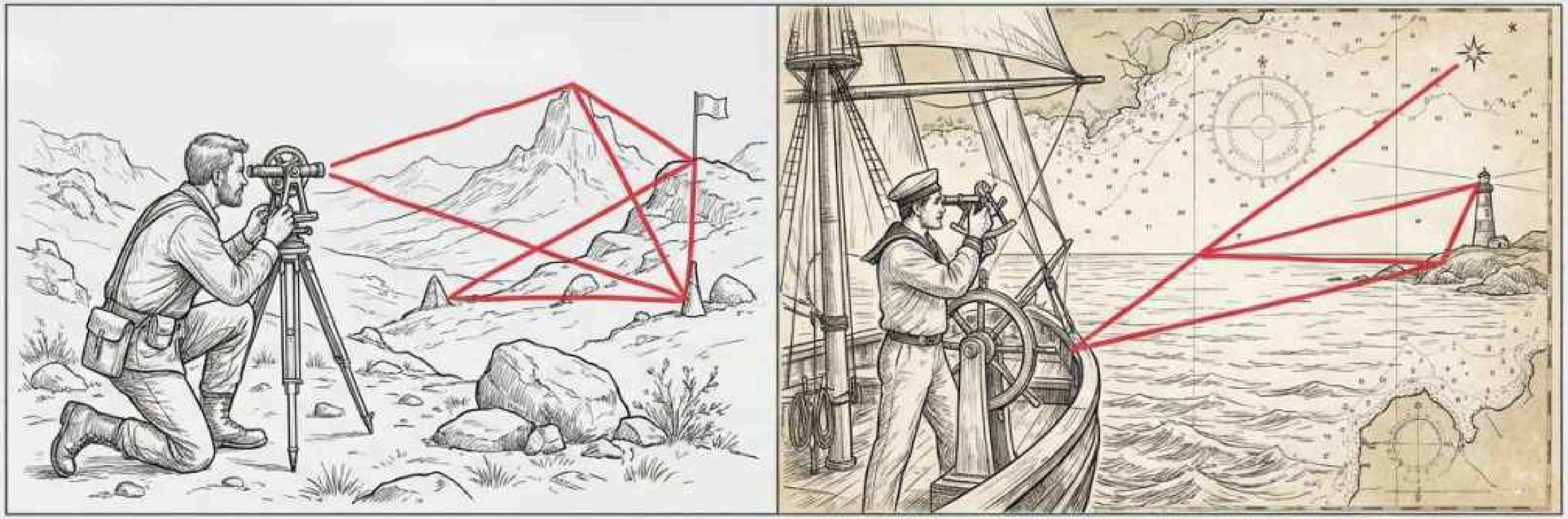
Lecture 6



The Laws of Sines and Cosines | The Ambiguous Case | Area & Heron's Formula | Ships, Surveying, & Choosing a Method

¶ 6.2

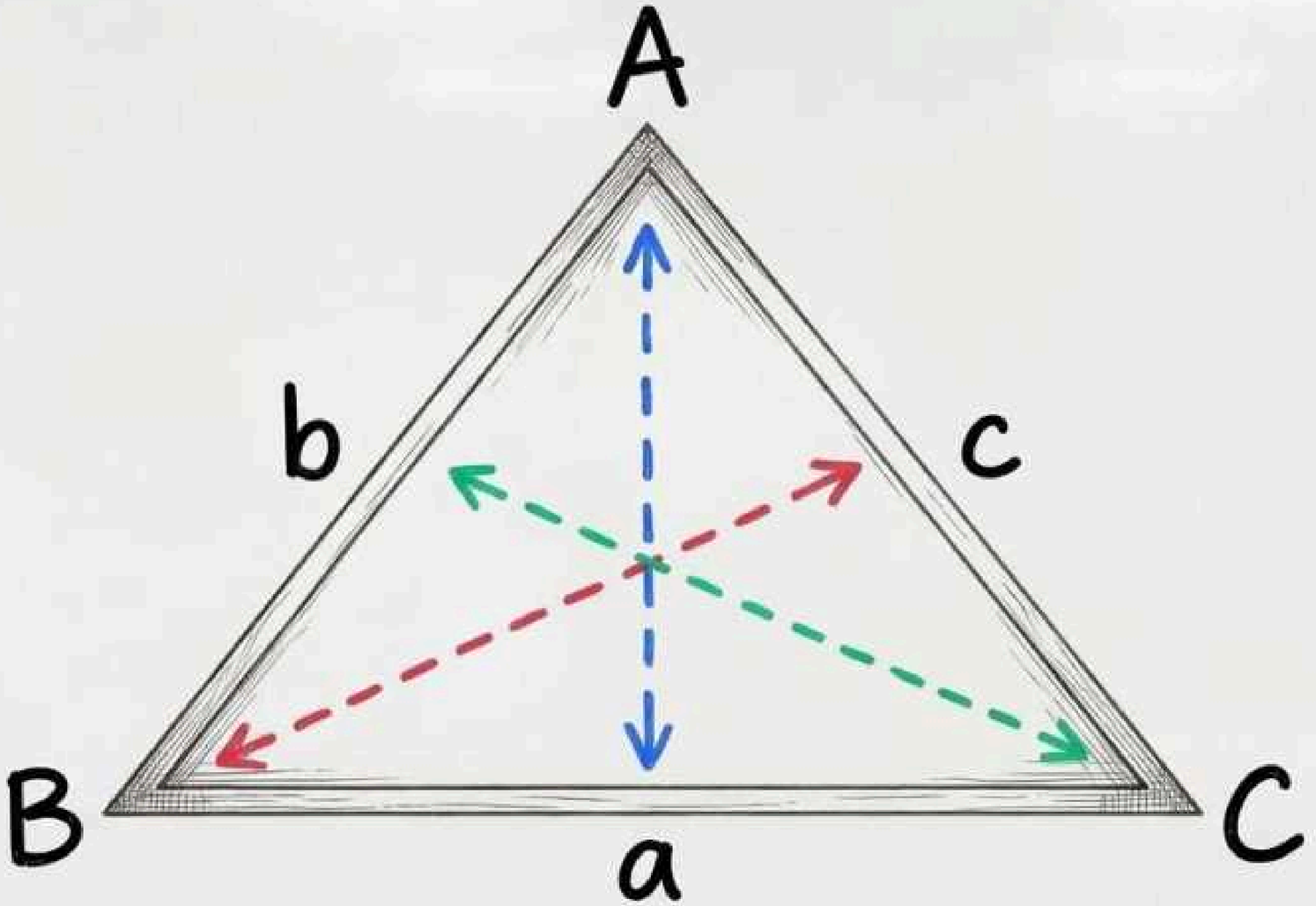
The real world rarely provides right angles.



Surveyors, sailors, and pilots constantly encounter triangles of every shape. This lecture introduces **two laws** to solve them, plus methods to **find area** without needing a triangle's height.

The Universal Blueprint: Naming Angles and Sides

¶ 6.3

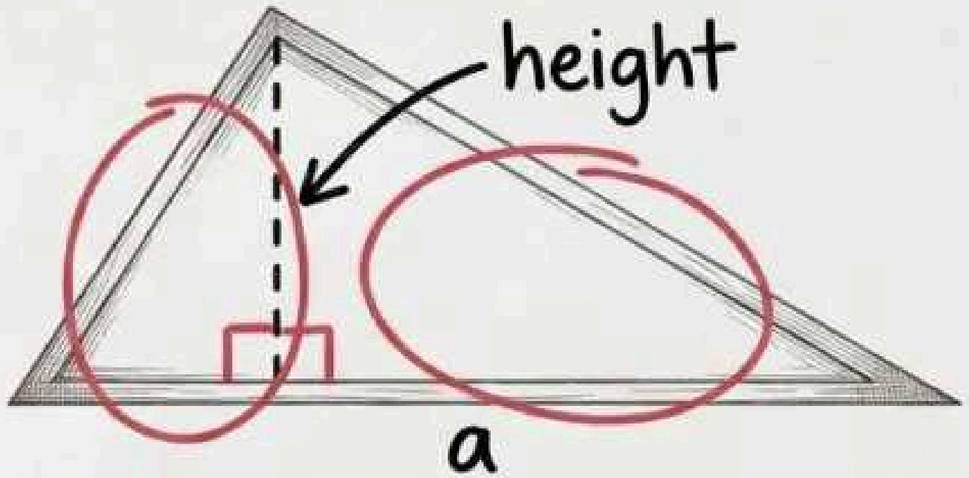


Keep to this convention, and the formulas will be easy to apply.

¶ 6.4

The Law of Sines: Equating the Ratios

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

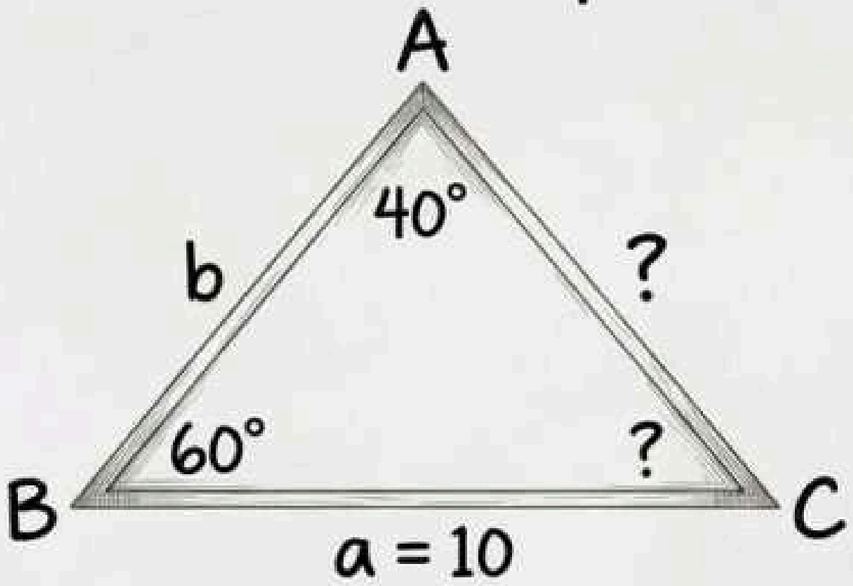


The ratio of a side to the sine of its opposite angle is identical for all three sides. It follows directly from comparing the heights of the two embedded right triangles.

¶6.5

Applying the Law of Sines (Known: Two Angles, One Side)

The Setup



The Math

Step 1 (Angle C)

$$180° - 40° - 60° = \boxed{80°}$$

Step 2 (Side b)

$$b = \frac{10 \times \sin 60°}{\sin 40°} \approx \underline{13.47}$$

Rule: Use when you know two angles and a side, or two sides and a non-included angle.

¶ 6.6

Solving the Triangle & The Quick Test

$$c = \frac{10 \times \sin 80^\circ}{\sin 40^\circ} \approx 15.32$$



The order of angles must match the order of sides. The largest side is always opposite the largest angle, and the smallest opposite the smallest. This is a quick test for any answer.

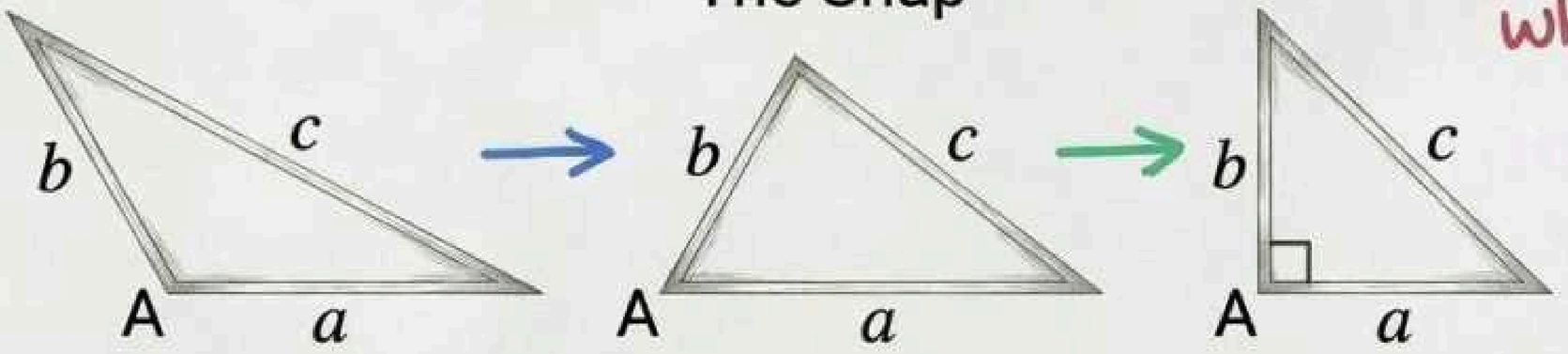
¶6.7

The Law of Cosines: The Generalized Pythagorizan Theorem

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Zeroes out when A = 90°

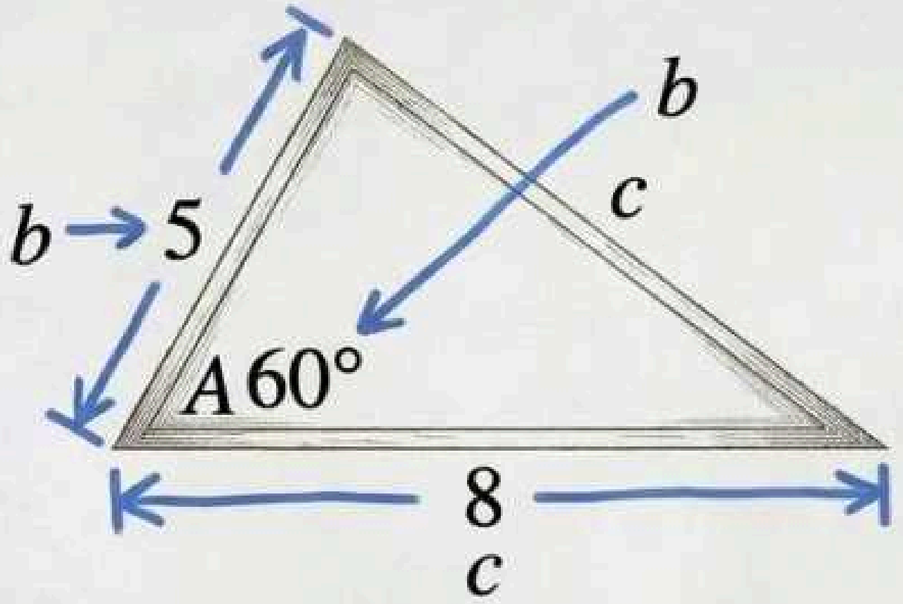
The Snap



If angle A is 90°, the cosine is zero, and we get back the Pythagorean theorem exactly. The extra term simply corrects for the angle not being a right angle.

¶ 6.8

Applying the Law of Cosines (Known: Two Sides, Included Angle)



$$a^2 = 25 + 64 - 2(5)(8)\left(\frac{1}{2}\right)$$

$$a^2 = 49$$

$$a = 7$$

Rule: Use it when you know two sides and the angle between them, or all three sides.

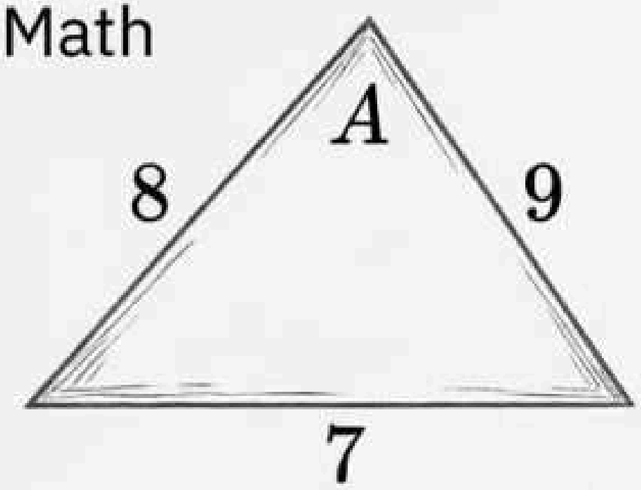
¶ 6.9

Rearranging for Three Known Sides (Finding Angle A)

Algebraic Transformation

$$a^2 = b^2 + c^2 - 2bc\cos A \longrightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Applied Math



$$\cos A = \frac{64 + 81 - 49}{144}$$

$$\cos A = \frac{96}{144} = \frac{2}{3} \qquad A \approx 48.19^\circ$$

When you only have sides (e.g., 7, 8, and 9), isolate the cosine to find the first angle.

¶ 6.10

Completing the Puzzle (Finding Angles B and C)

Finding B:

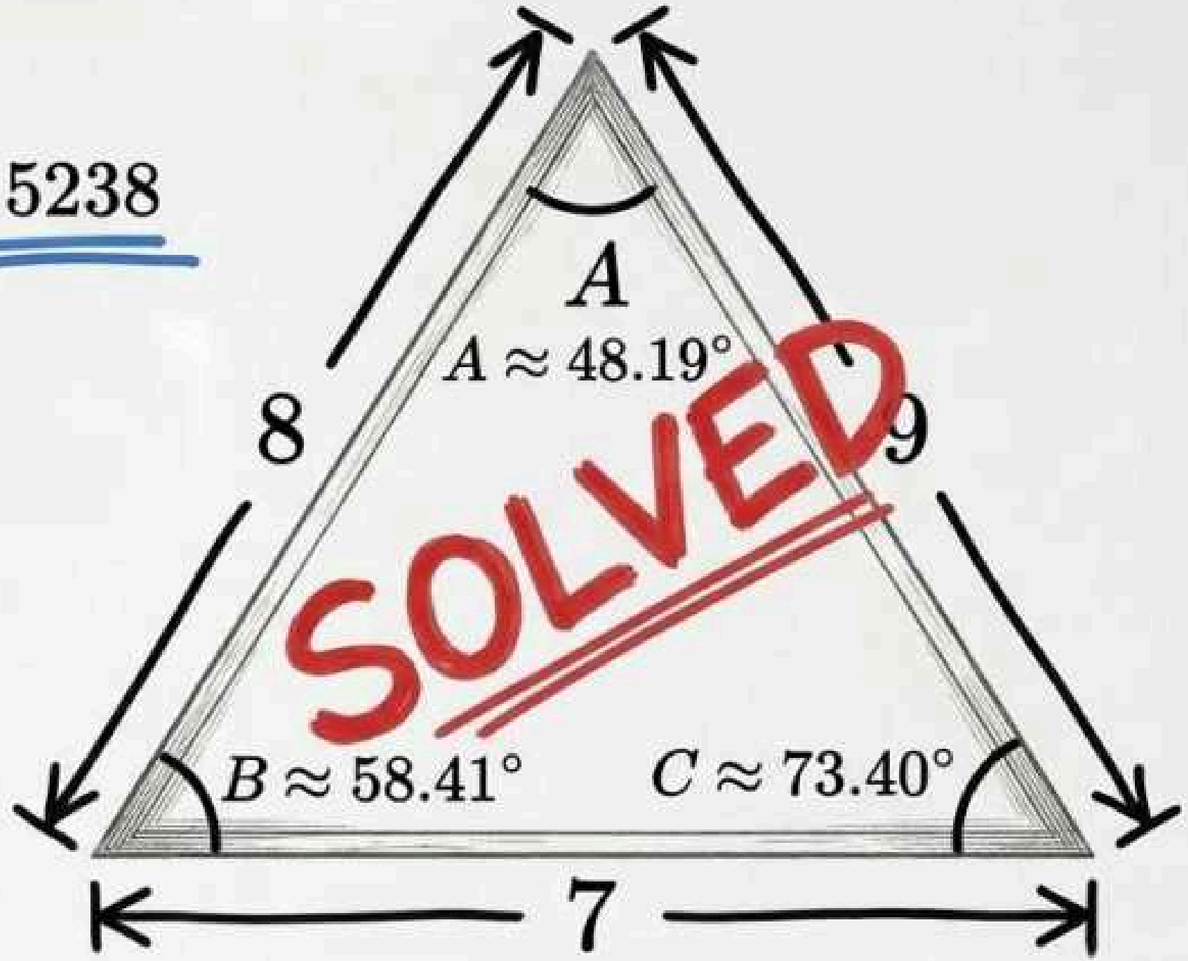
$$\cos B = \frac{49 + 81 - 64}{2(7)(9)} = \frac{66}{126} \approx 0.5238$$

$$B \approx 58.41^\circ$$

Finding C:

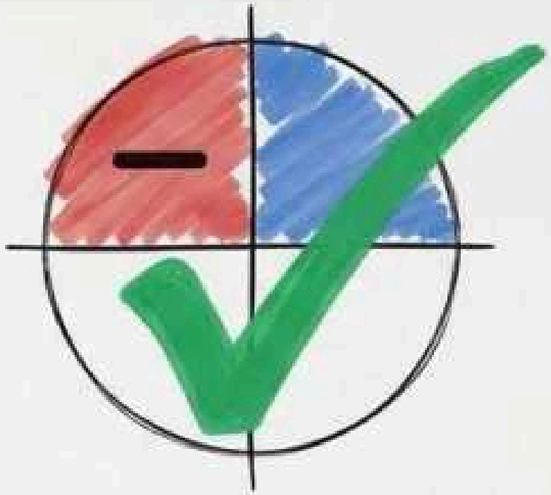
The rest → $C \approx 73.40^\circ$

Find the angle opposite the 8 in the same way. The final angle, C, is simply whatever remains to reach 180 degrees.



¶ 6.11

The Cosine Safety Rule

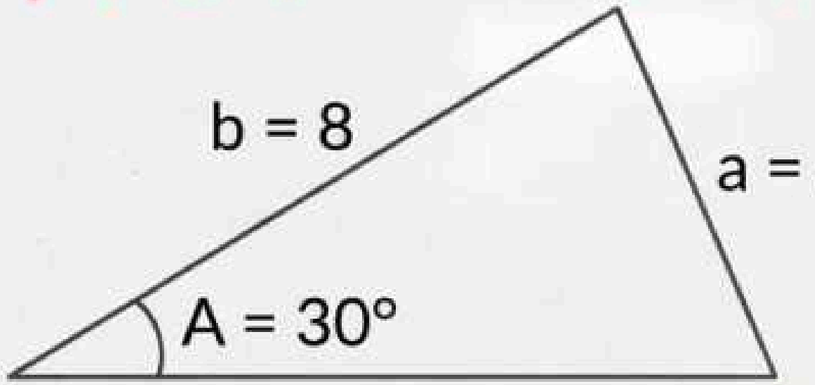
The Danger of Sine	The Safety of Cosine
	
Sine: Positive in both.	Cosine: Distinctly negative for obtuse angles.

When the Law of Cosines finds an angle, the answer is unambiguous because cosine yields negative values for obtuse angles.

Pro Tip: When in doubt, use the law of cosines for the first angle, and then either law for the others.

The Ambiguous Case: A Mathematical Trap



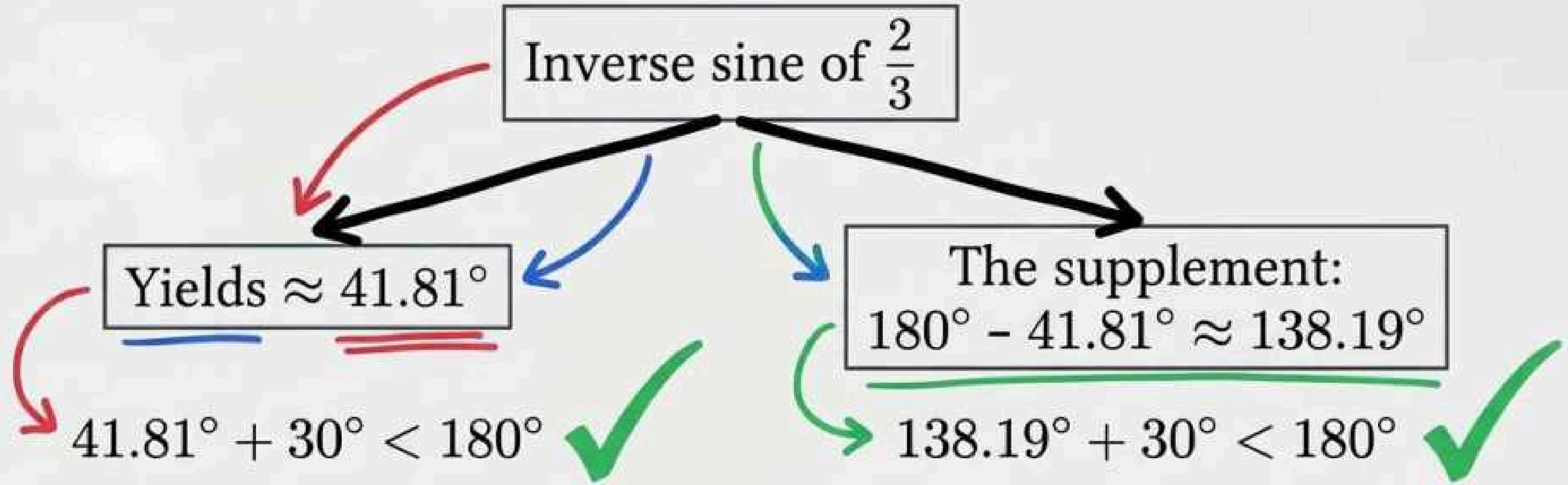


$$\sin B = \frac{8 \times \frac{1}{2}}{6} = \frac{2}{3}$$

Suppose you know two sides and an angle opposite one of them.
There may be zero, one, or two valid triangles hidden in the math.

¶ 6.13

One Sine, Two Valid Angles



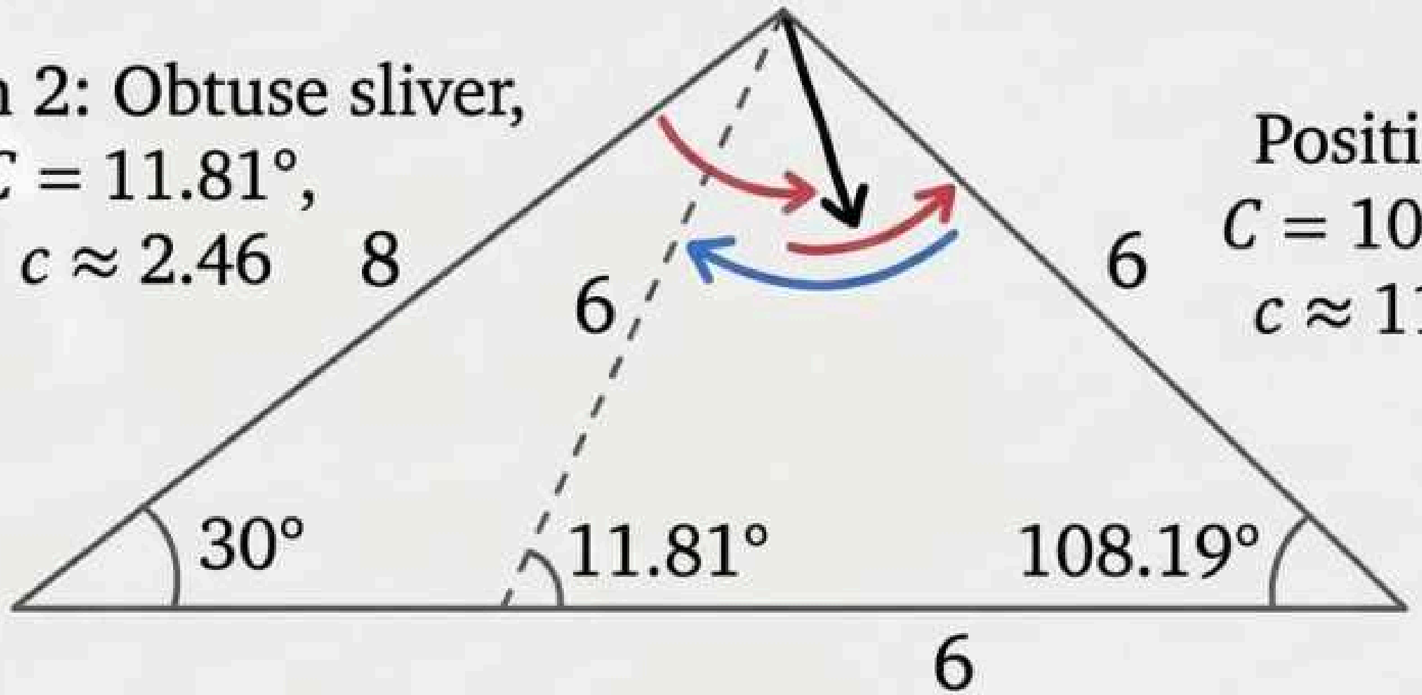
Both the primary angle and its supplement share the exact same sine value. Furthermore, both fit within a triangle alongside the starting $30°$ angle without exceeding the $180°$ limit.

¶ 6.14

Two Triangles Revealed

Position 2: Obtuse sliver,
$C = 11.81°$,
$c \approx 2.46$

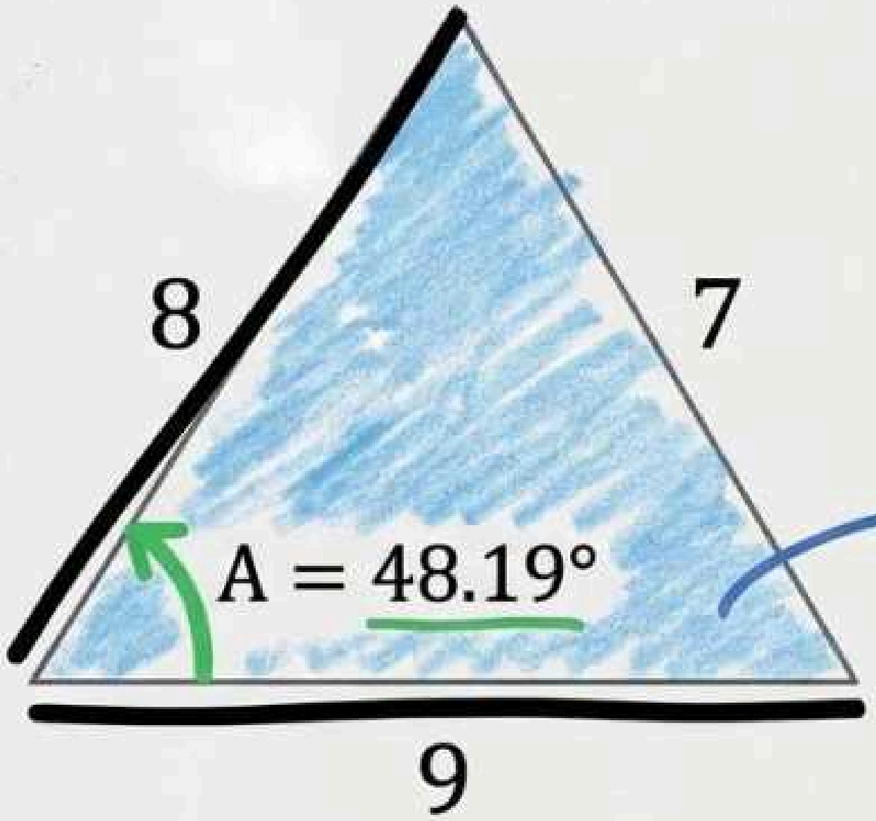
Position 1: Acute,
$C = 108.19°$,
$c \approx 11.4$



Whenever the sine calculates to a value above 1, there is no triangle at all.

¶ 6.15

Calculating Area Without Height (Via Sine)



$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2} \times 8 \times 9 \times 0.7454$$

$$\text{Area} \approx 26.83$$

If you know two sides and the angle between them, you can bypass the need to drop a perpendicular height line.

¶ 6.16

The Ultimate Check: Heron's Formula

s = half the perimeter = 12

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{12 \times 5 \times 4 \times 3}$$

$$= \sqrt{720}$$

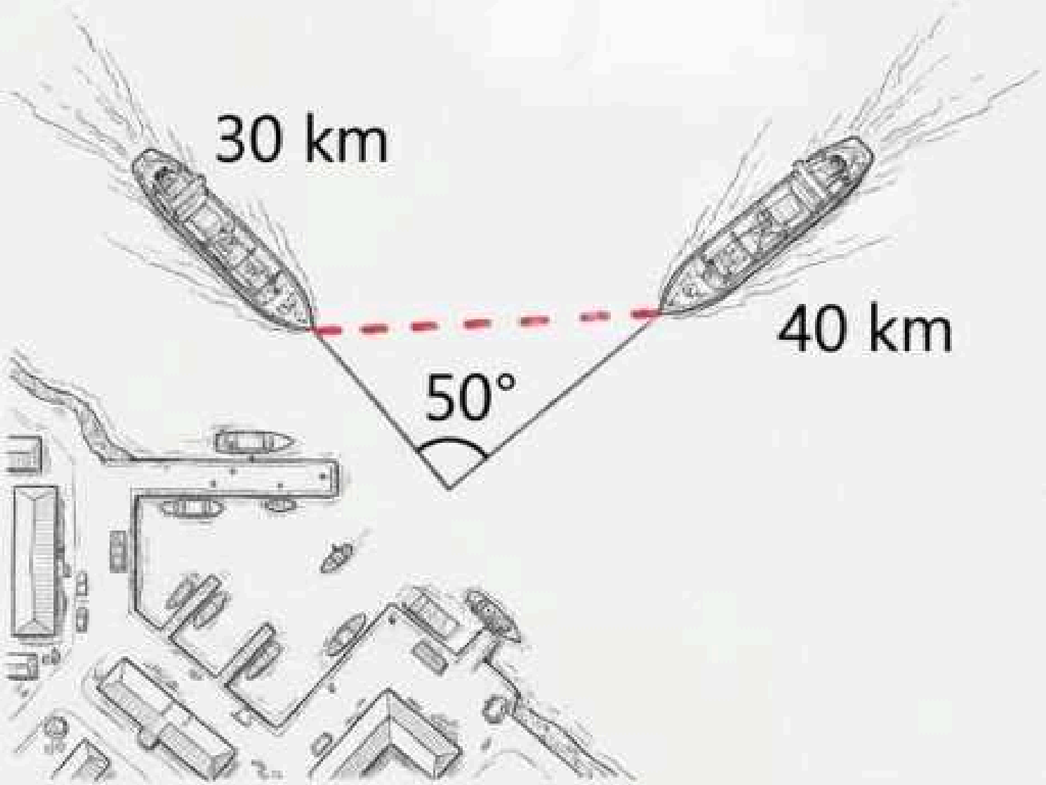
$$\approx 26.83$$

✓ The two methods agree.

Heron's formula relies exclusively on the lengths of the sides, providing an independent verification of our previous area calculation.

¶ 6.17

Application in the Wild: Nautical Navigation



$$d^2 = 900 + 1600 - 2(30)(40)\cos 50^\circ$$

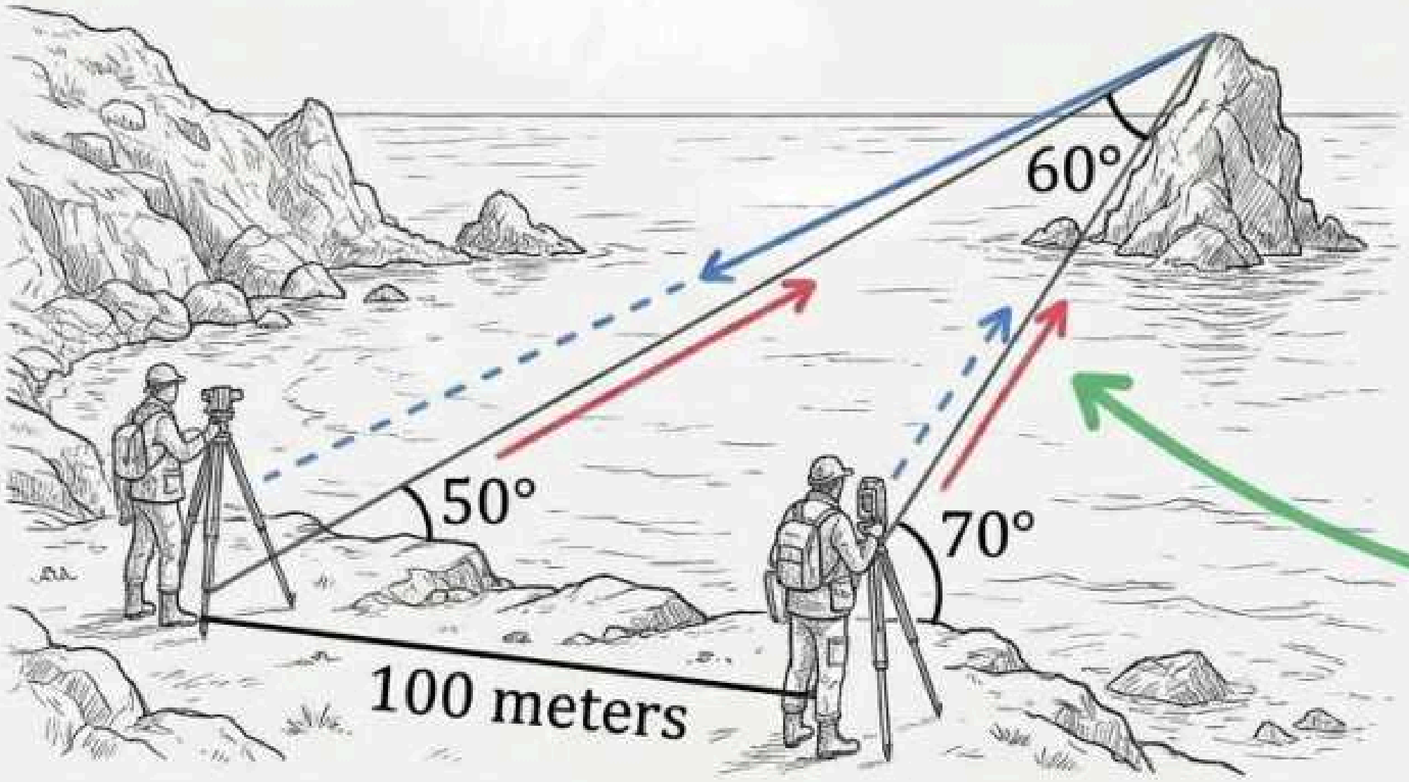
$$d^2 \approx 957$$

$$\text{Distance} \approx 30.94 \text{ km}$$

Two ships leave port on diverging paths. The Law of Cosines easily calculates the exact distance between them.

¶6.18

Application in the Wild: Surveying



Calculating distance from Observer 1 to the rock:

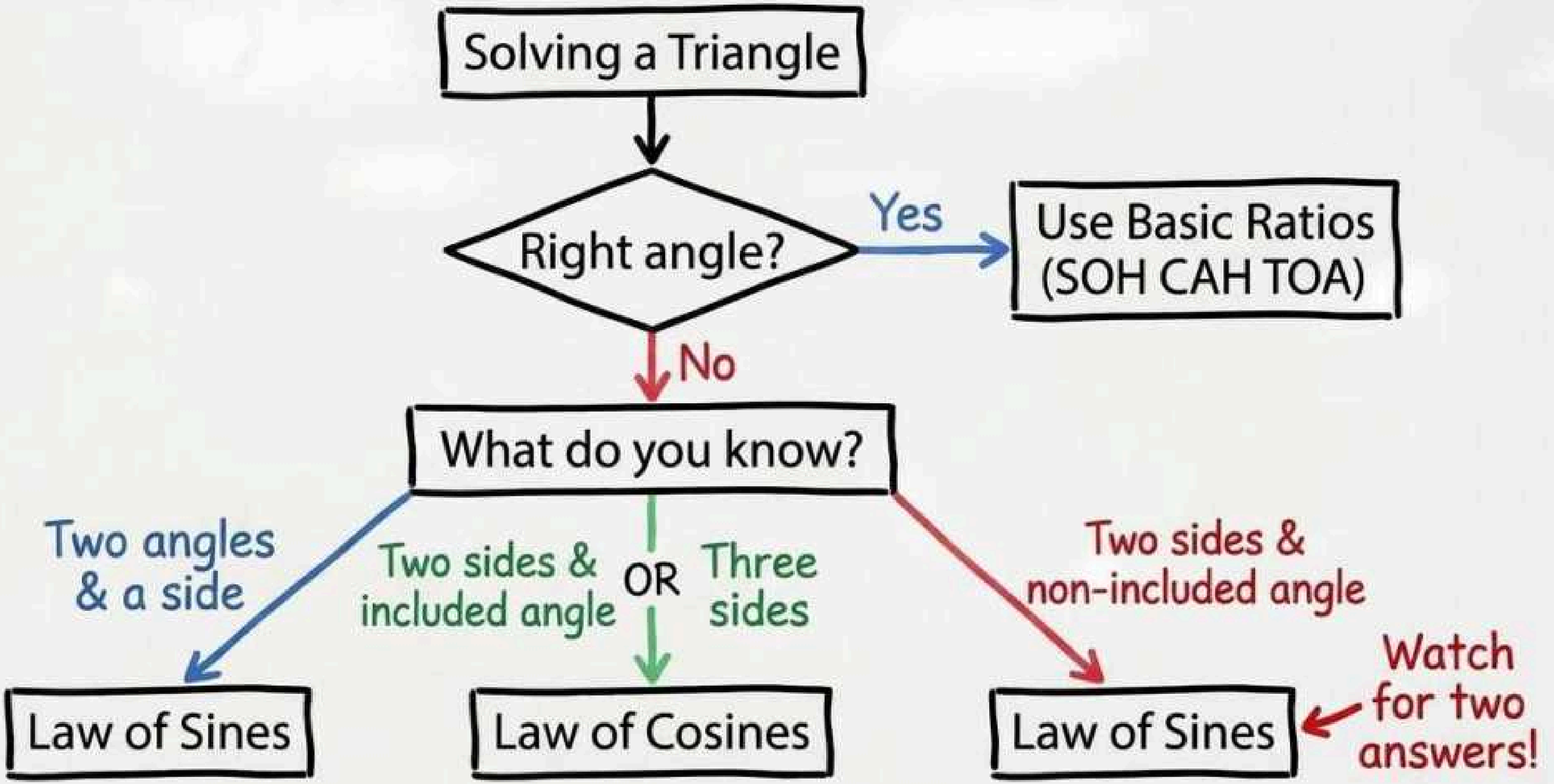
$$\text{Distance} = 100 \times \frac{\sin 70°}{\sin 60°}$$

$$\approx 108.5 \text{ meters}$$

With a known baseline on shore and two angles of observation, the Law of Sines pinpoints the distance to unreachable offshore landmarks.

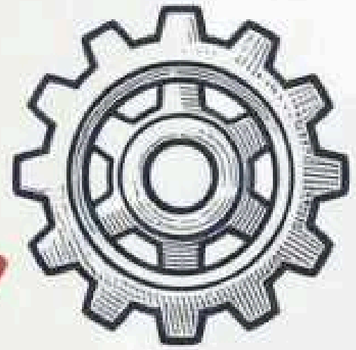
¶ 6.19

The Master Decision Matrix: Choosing Your Method

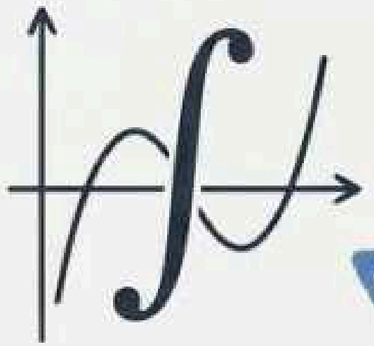


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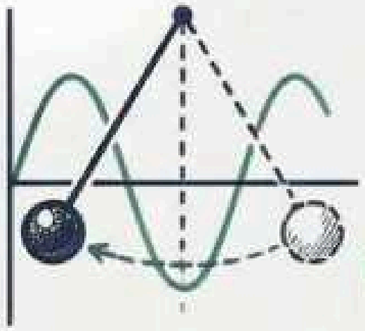
The Foundations of Trigonometry are Complete



Engineering



Calculus



Physics

You can now compute ratios, solve right triangles, use radians, graph waves, apply identities, and solve any triangle.
These are the tools that drive calculus, physics, and engineering.
Practice with the quizzes and tell us what you build next.

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