Polynomial Functions: Lecture Overview

¶ 3.1

Polynomial Functions

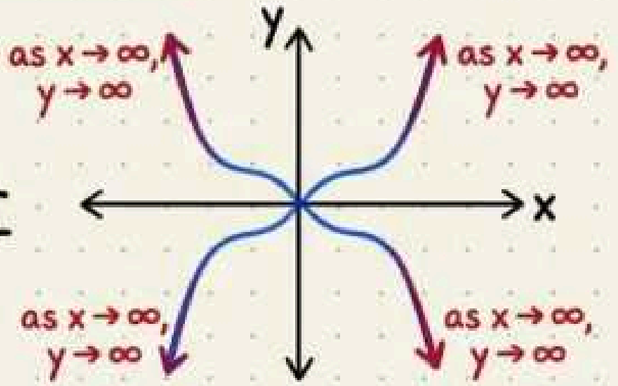
Anatomy

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Exponents (Degrees)

Coefficients (Constants)

End Behavior



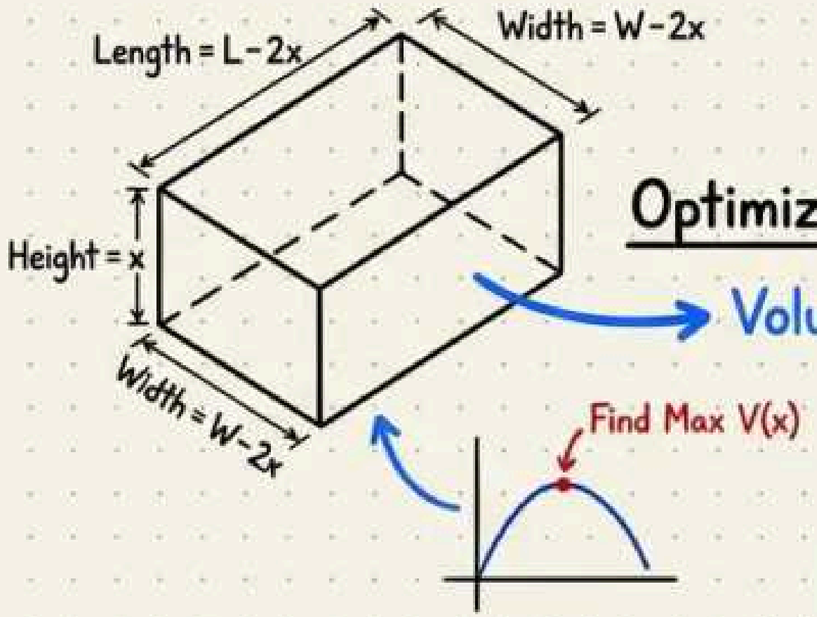
Finding Zeros

Synthetic division:

c	a_n	a_{n-1}	⋯	a_1	a_0
		c	c	⋯	c
	$c + a_{n-1}$	⋯	⋯		$-a_0$

Remainder = $P(c)$ ← if Remainder = 0, then (x−c) is a factor

Optimization



Volume $V(x) = x(L-2x)(W-2x)$

Complex Numbers



$i = \sqrt{-1}$

$i^2 = -1$

Essential for solving all polynomials

The most important family of functions in algebra. Take it slowly: these techniques return in every later course.

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Anatomy of a Polynomial

¶ 3.2

Degree (Highest Power = 3)

$$f(x) = 2x^3 - 5x^2 + x - 7$$

Leading Coefficient
(The multiplier of the highest power = 2)

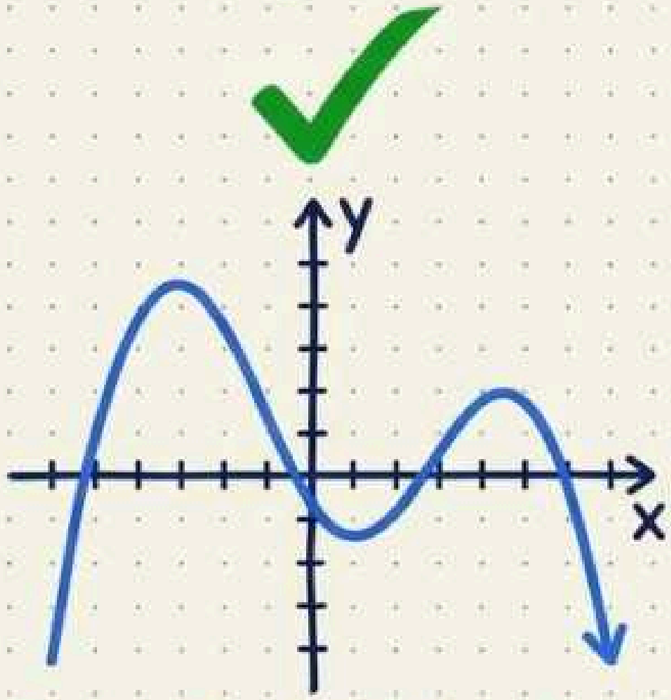
Constant Term
(The standalone number = -7)

Definition: A sum of terms, each a number times a whole-number power of x.

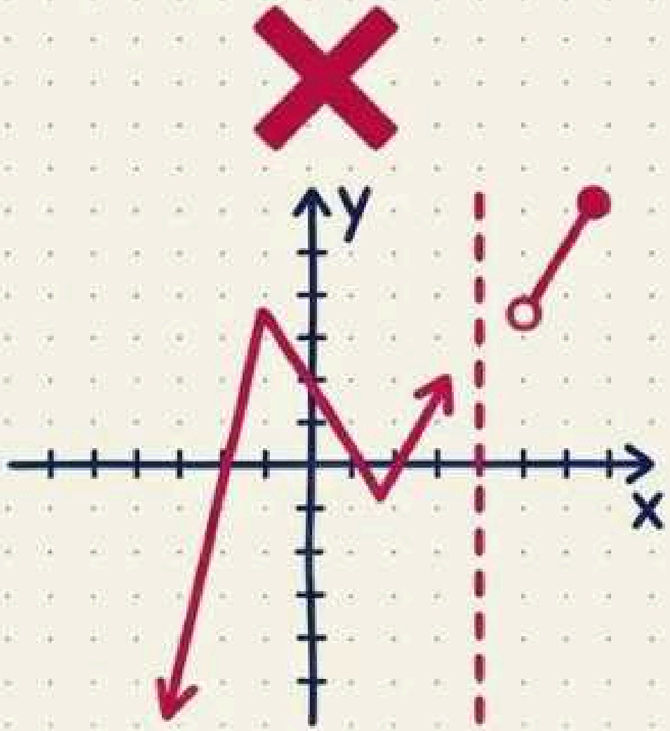
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The Shape of Polynomial Graphs

¶ 3.3



Smooth & Unbroken
(No gaps, jumps, or sharp corners).



Not a Polynomial.

For a polynomial of degree n:

Max Turning Points = n - 1

Max x-axis Crossings = n

Examples: Line = 0 turns, Parabola = 1 turn, Cubic = up to 2 turns.

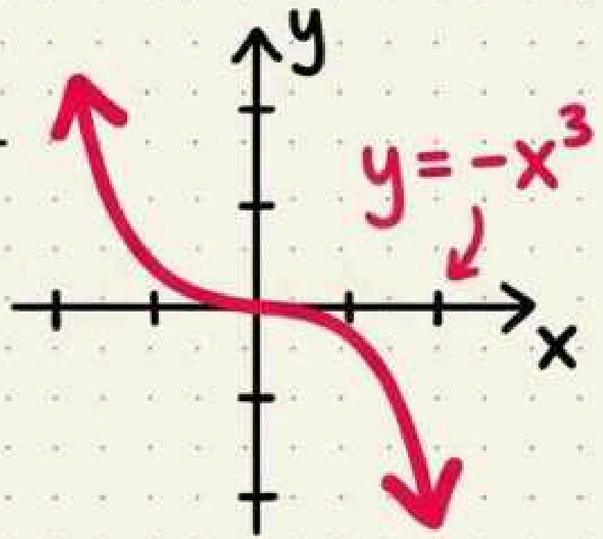
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End Behavior: The Outer Limits

¶ 3.4

Only the Leading Term matters for the far left/right.

	Even Degree	Odd Degree
+ Leading Coeff	↑ ↑	↓ ↓
− Leading Coeff	↓ ↑	↑ ↓



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The Factor Theorem

¶ 3.5

$(x - c)$ is a factor



c is a zero

$$f(x) = (x+2)(x-1)(x-4)$$

$x = -2$ $x = 1$ $x = 4$

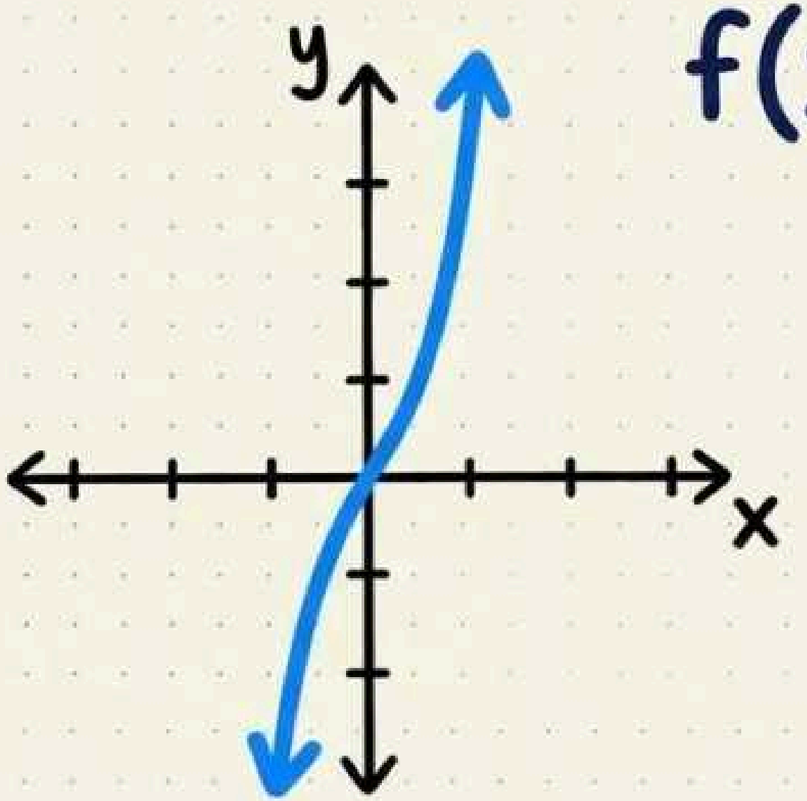
✓ Going from zeros to a formula is just as easy.

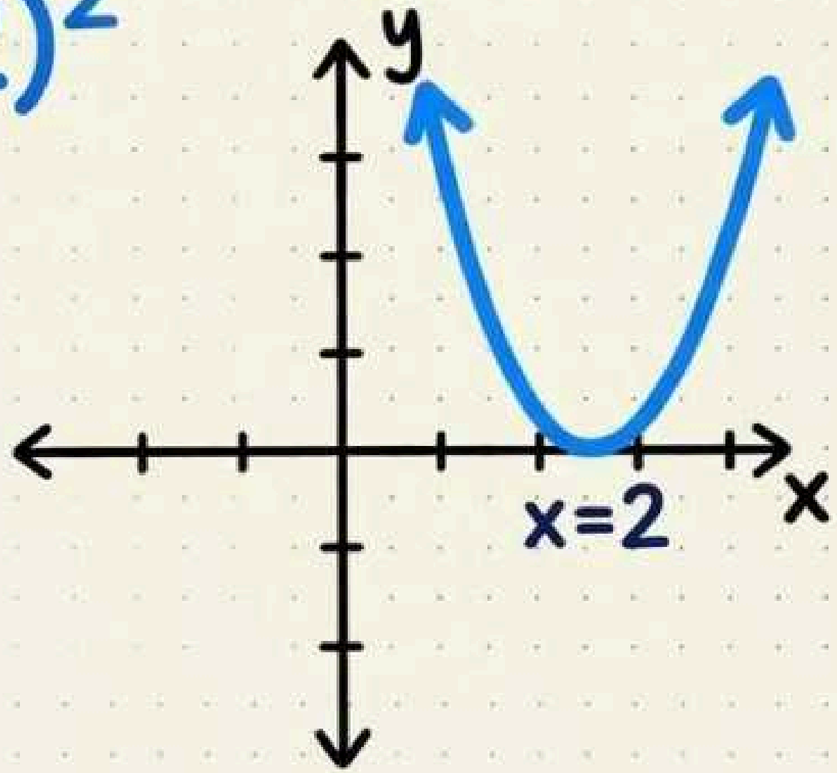
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Multiplicity: Crossing vs. Touching

¶ 3.6

$$f(x) = x(x-2)^2$$





Odd Multiplicity (Power of 1)

→ Graph CROSSES.

Even Multiplicity (Power of 2)

→ Graph TOUCHES and turns.

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Sketching a Polynomial: The Setup

¶ 3.7

$$f(x) = (x+2)(x-1)(x-4)$$

Step 1: Y-intercept.

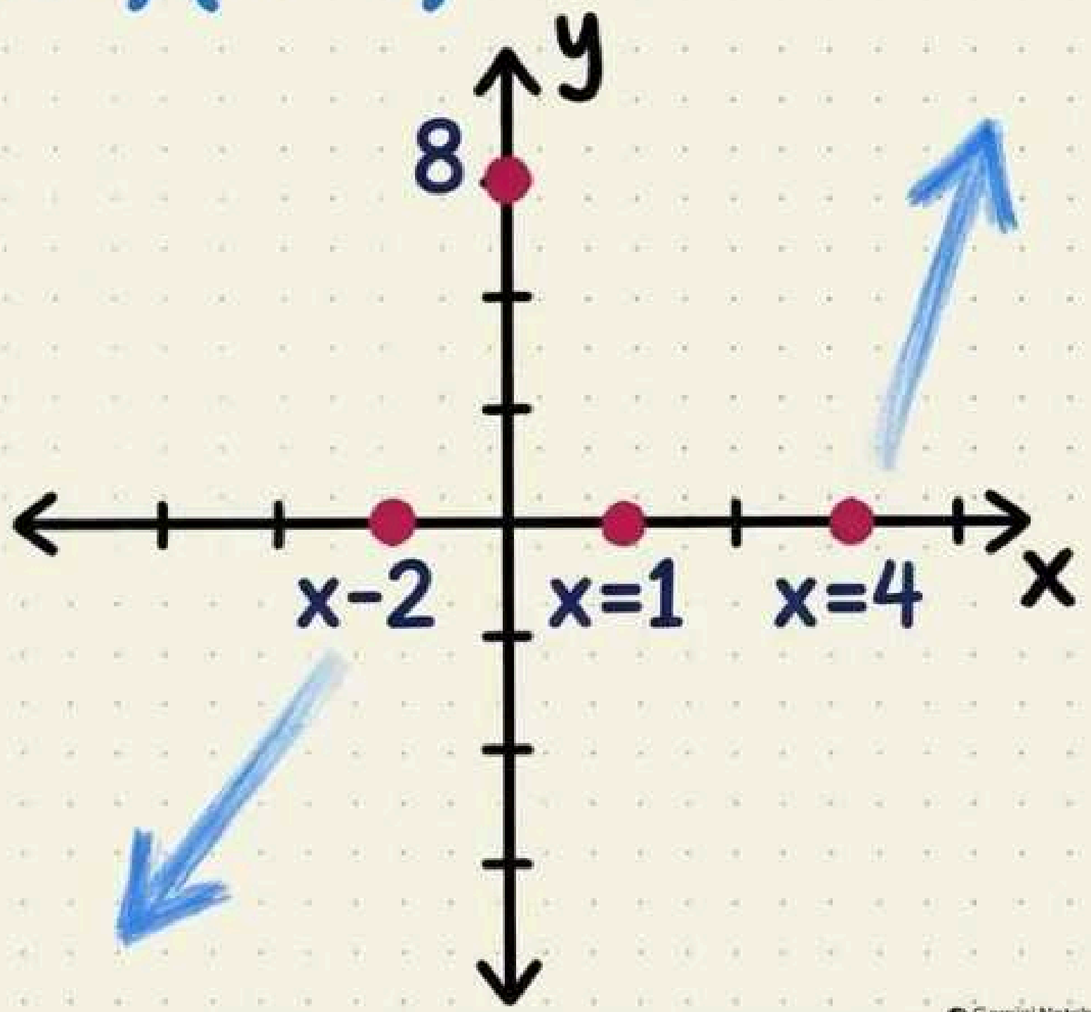
$f(0) = 2 \times -1 \times -4 = 8.$

Step 2: Zeros.

Set factors to zero.

Step 3: End Behavior.

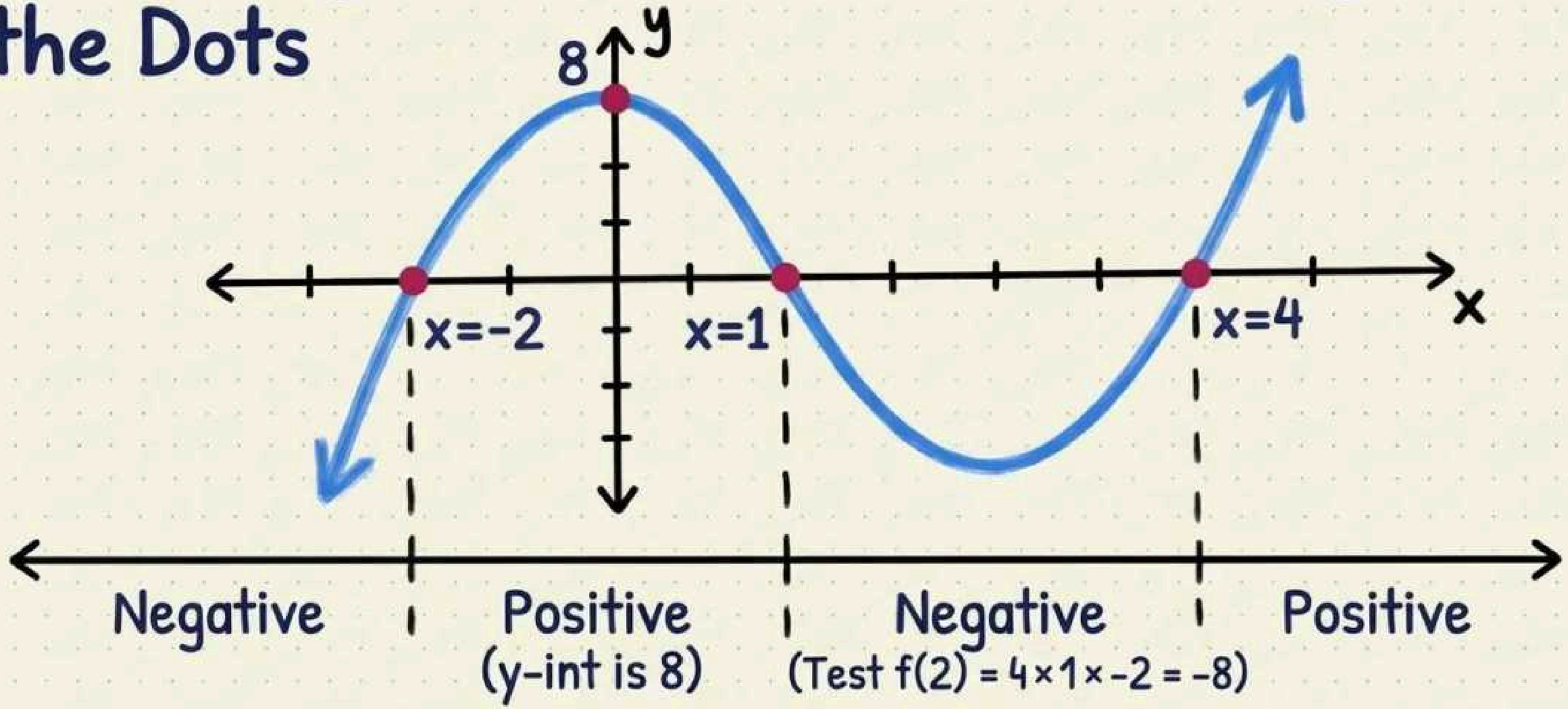
Odd degree + Positive leading coefficient.



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Sketching a Polynomial: Connecting the Dots

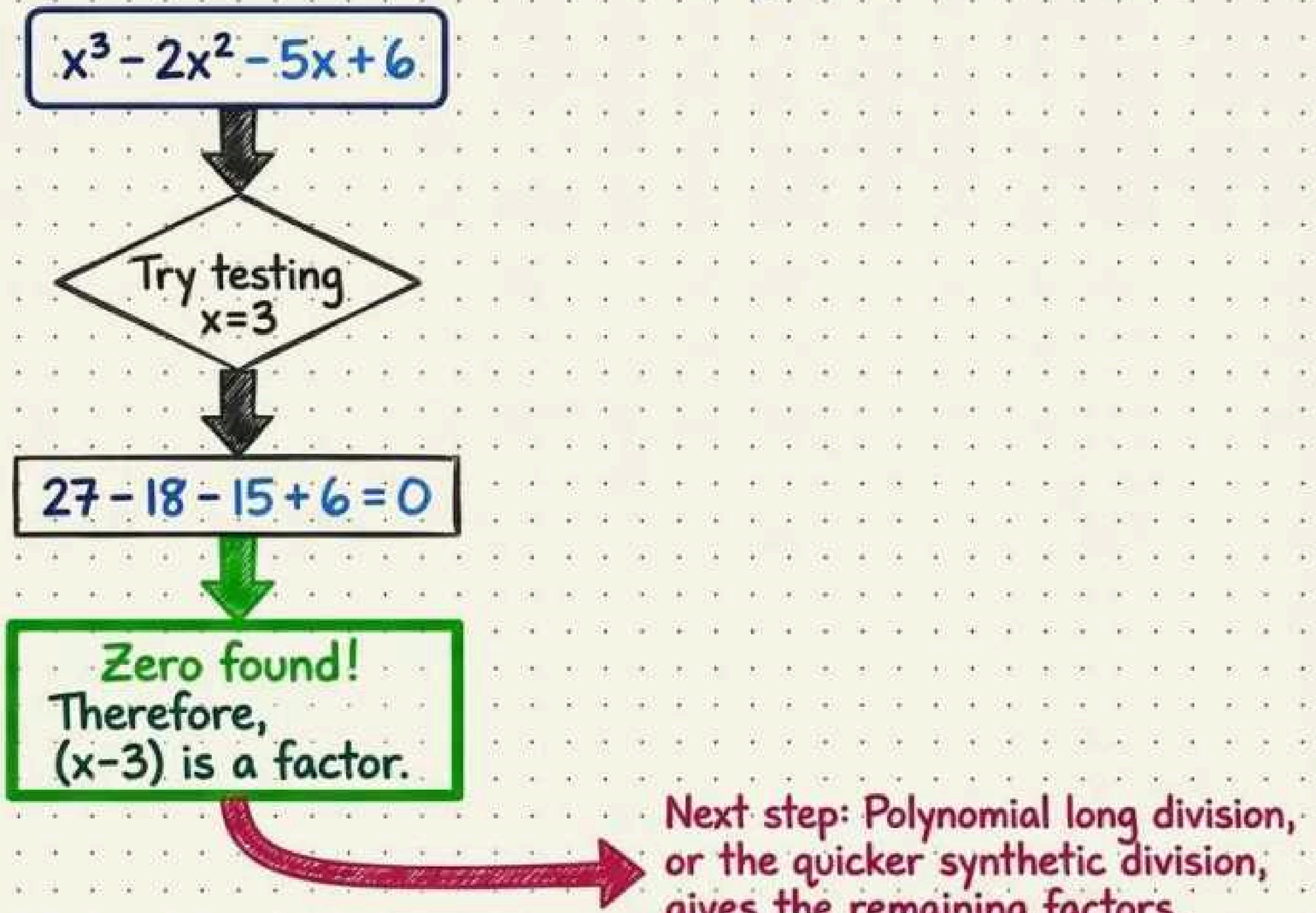
¶ 3.8



The curve rises, falls, then rises smoothly through the plotted dots.

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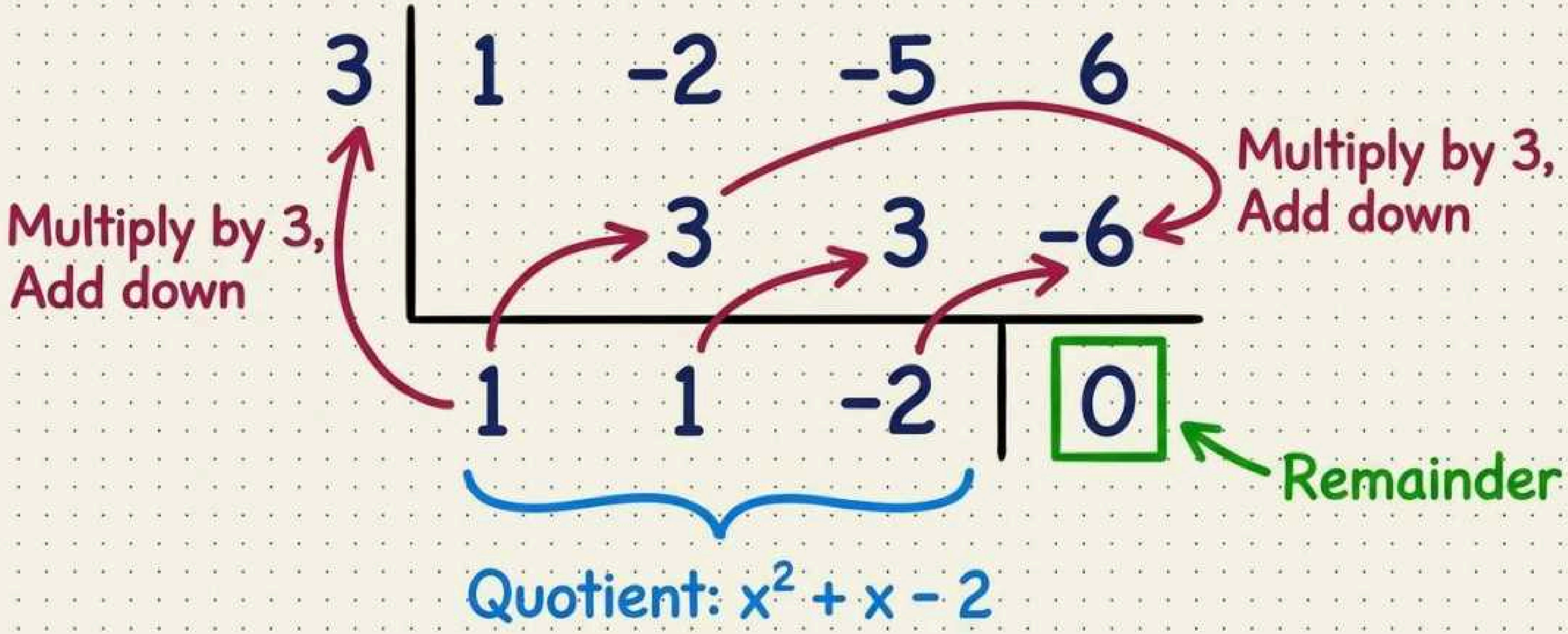
Finding Zeros of Harder Polynomials



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Synthetic Division Process

¶ 3.10



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The Remainder Theorem

¶ 3.11

Direct Evaluation

$$f(2) = (2)^3 - 2(2)^2 - 5(2) + 6$$

$$f(2) = 8 - 8 - 10 + 6$$

$$f(2) = \boxed{-4}$$

Synthetic Division

2	1	-2	-5	6
		2	0	-10
	1	0	-5	$\boxed{-4}$

Dividing a polynomial by $(x-c)$ leaves a remainder exactly equal to $f(c)$. This provides a fast alternative way to evaluate functions.

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The Rational Root Theorem

¶3.12

How do you find the first zero? Narrow the search.

$$1x^3 - 2x^2 - 5x + 6$$

Candidates: ± 1. $\}$ q

p $\{$ Candidates: $\pm 1, 2, 3, 6$. $]$

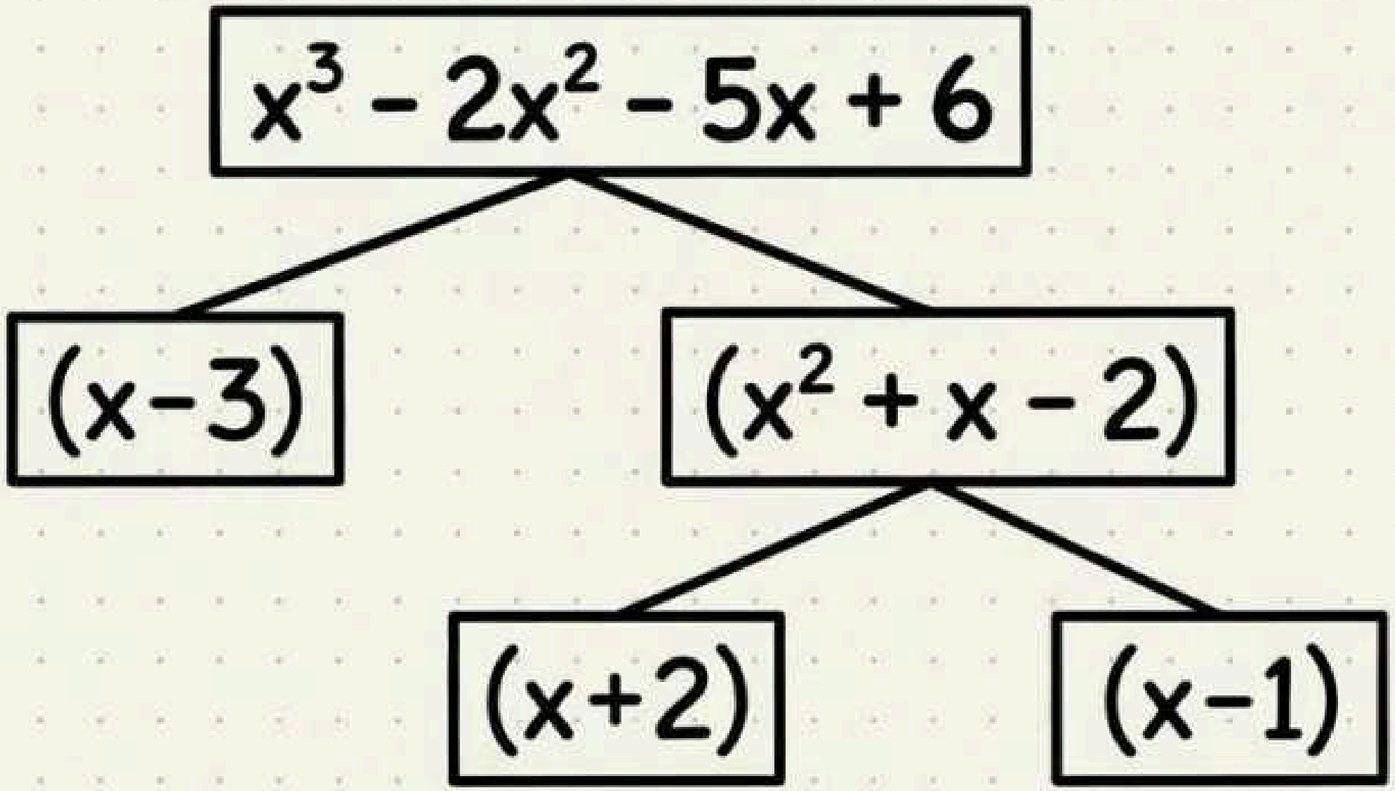
$$\text{Roots} = \frac{p}{q}$$

Testing the first candidate x=1 gives
$1 - 2 - 5 + 6 = 0$. (Zero found!)

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Factoring Completely

¶ 3.13



$f(x) = (x-3)(x+2)(x-1)$ Zeros: 3, -2, 1.

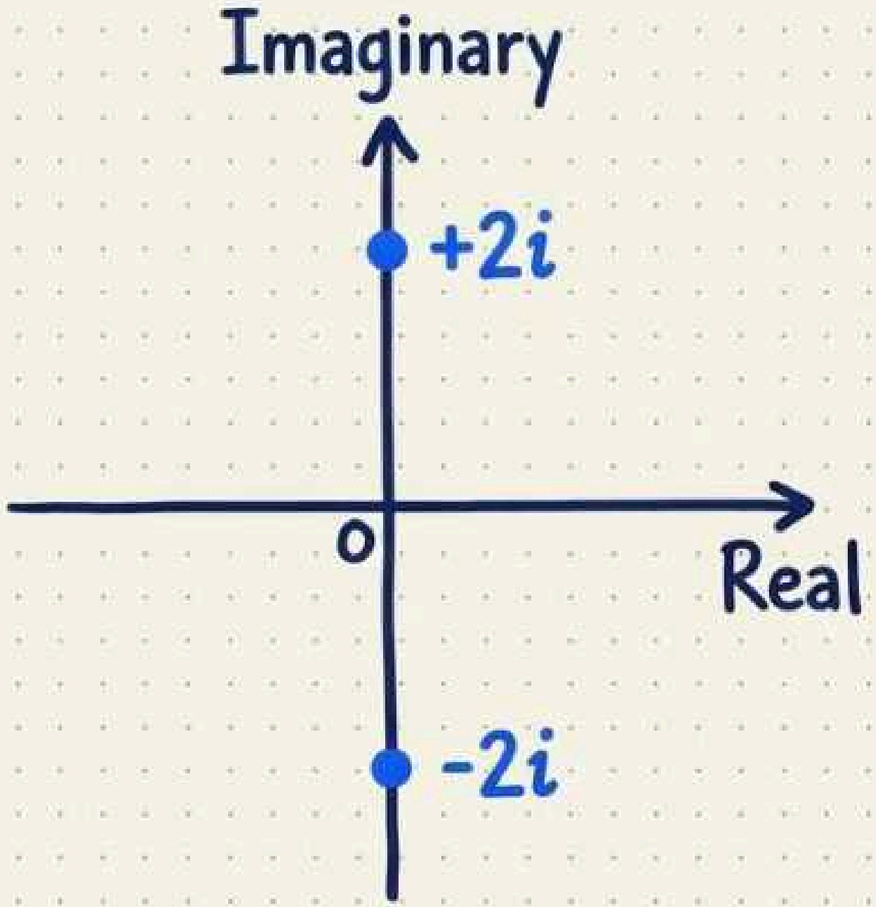
Another example: $2x^3 - 3x^2 - 3x + 2$ factors to yield zeros 2, $\frac{1}{2}$, and -1

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Expanding to Complex Numbers

¶ 3.14

- **The Problem:** $x^2 + 4 = 0$ requires $x^2 = -4$. There are **no real zeros**.
- **The Solution:** Mathematicians introduce the **Imaginary Unit** i, defined by $i^2 = -1$.
- **The Result:** Solutions become $\pm 2i$.
- **Definition:** **Complex Numbers** = **Real part** + **Imaginary part**.



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The Quadratic Formula with Complex Roots

¶ 3.15

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$\sqrt{-16} \rightarrow 4i$

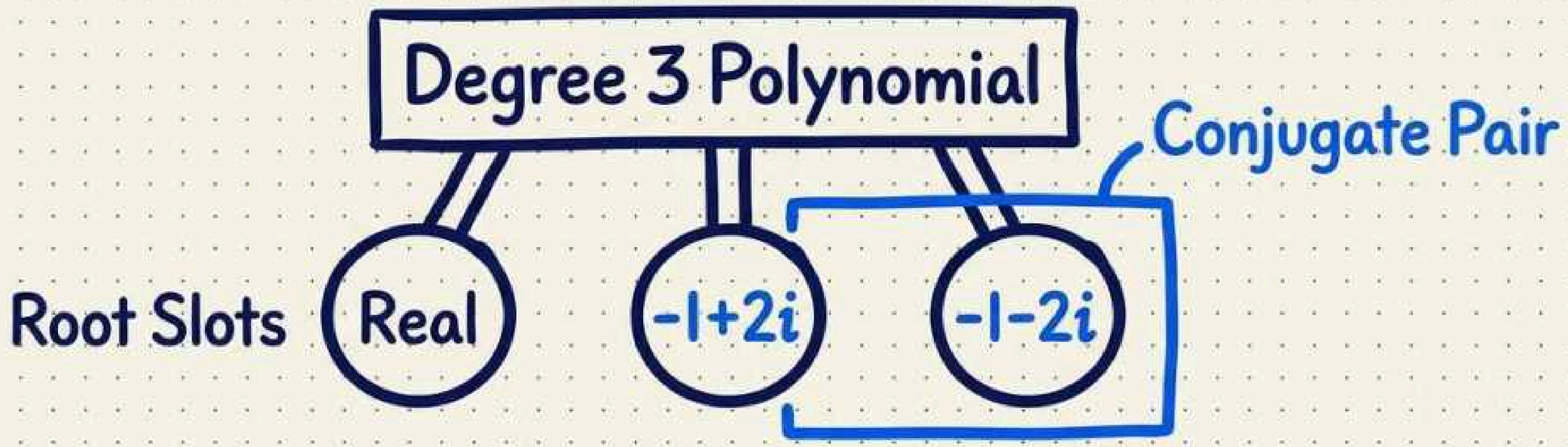
$$x = \frac{-2 \pm 4i}{2}$$

$$\checkmark\ x = -1 \pm 2i$$

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Fundamental Theorem of Algebra

¶ 3.16



- A polynomial of degree n has exactly n complex zeros (counting multiplicity).
- When coefficients are real, non-real zeros ALWAYS come in conjugate pairs.

Result: A real cubic always has at least one real zero, which is why every every cubic graph MUST cross the x-axis.

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Building a Polynomial from Zeros

¶ 3.17

Given: Zeros at 2 and −1 (multiplicity 2); y-intercept must be 6.

↓

Step 1 (Multiply Factors)

$f(x) = a(x-2)(x+1)^2$

↓

Step 2 (Evaluate)

Set $x=0 \rightarrow a(-2)(1)^2 = -2a$

↓

Step 3 (Adjust Scale)

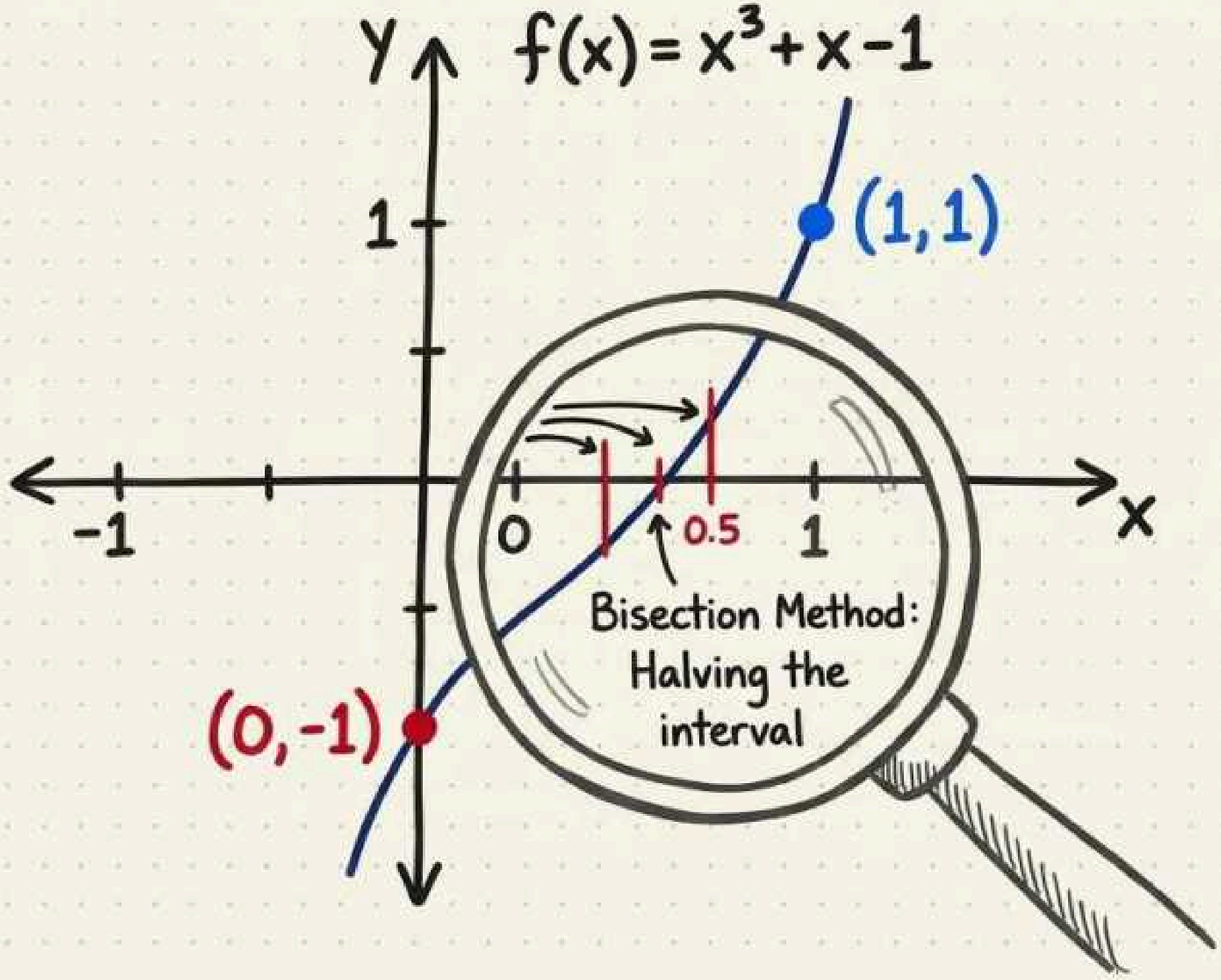
If $f(0)=6$, then $-2a=6 \rightarrow a=-3$

$$f(x) = -3(x-2)(x+1)^2$$

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Intermediate Value Theorem

¶ 3.18

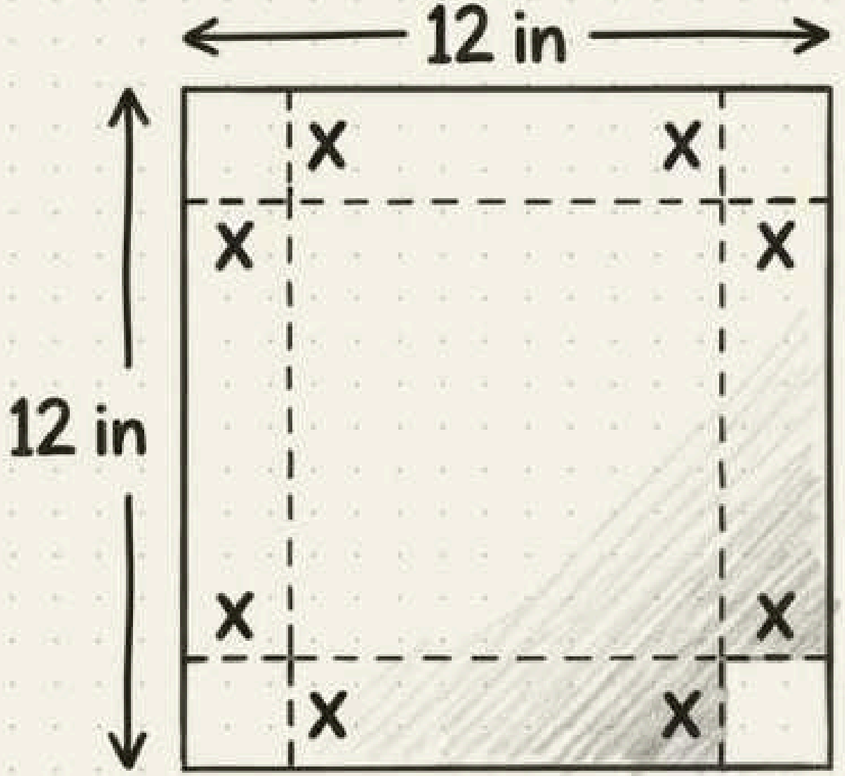


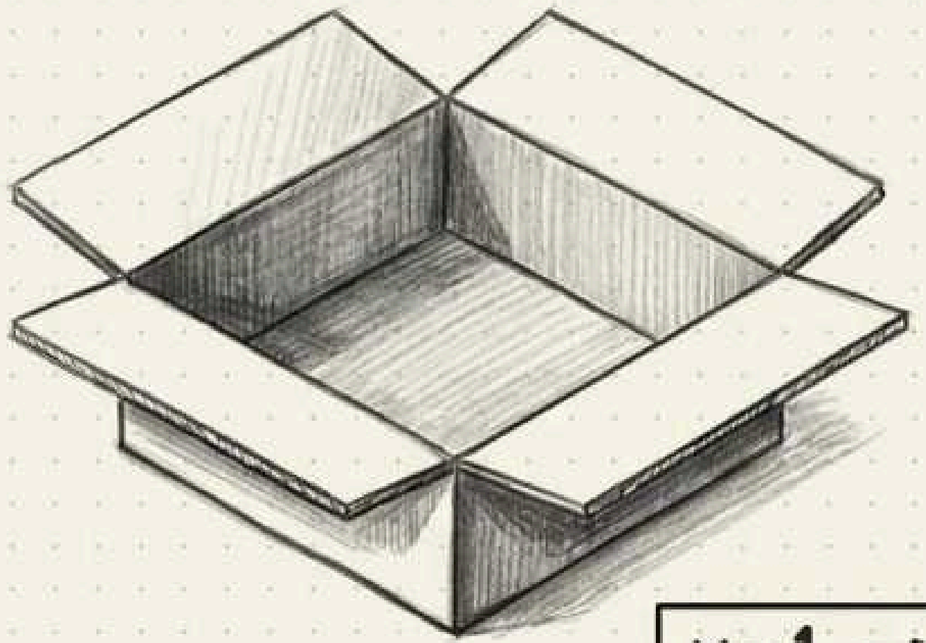
- Guarantees zeros for continuous functions: If the output changes sign, the graph MUST cross zero in between.
- By halving the interval again and again, we locate the root: ≈ 0.682.

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Application: Open-Box Volume Optimization

¶ 3.19





$$V = x(12-2x)^2$$

$x=1 \rightarrow V=100$	
$x=2 \rightarrow V=128$	
$x=3 \rightarrow V=108$	

The optimal cut to maximize volume is exactly 2 inches.

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Common Mistakes & Final Sketch

¶ 3.20

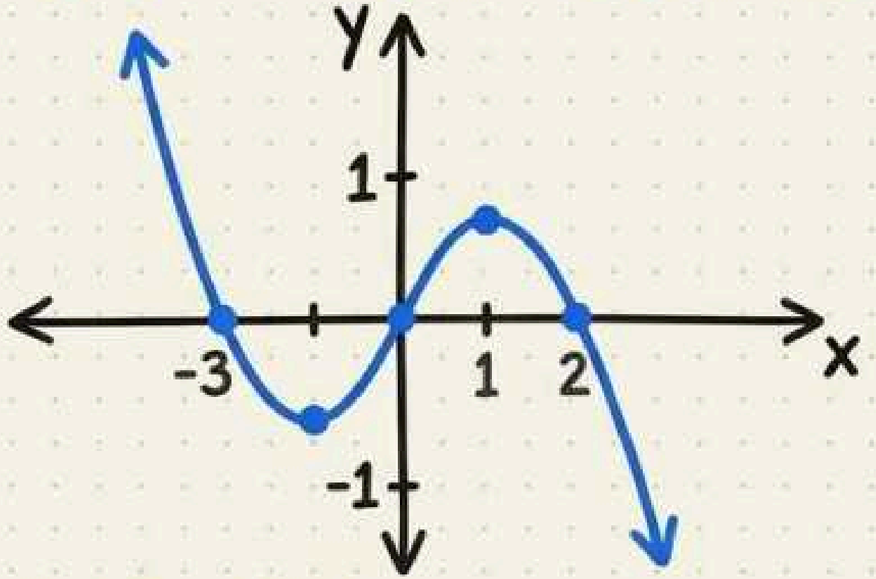
Common Pitfalls

Caution

- Forgetting Multiplicity
- Sign errors in Synthetic Division
- Wrong End Behavior for Odd Degree

Final Problem

$$f(x) = -x(x+3)(x-2)$$



Zeros: 0, -3, 2

Y-intercept: 0

End behavior: Up/Down

Next Lecture: Rational Functions.

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