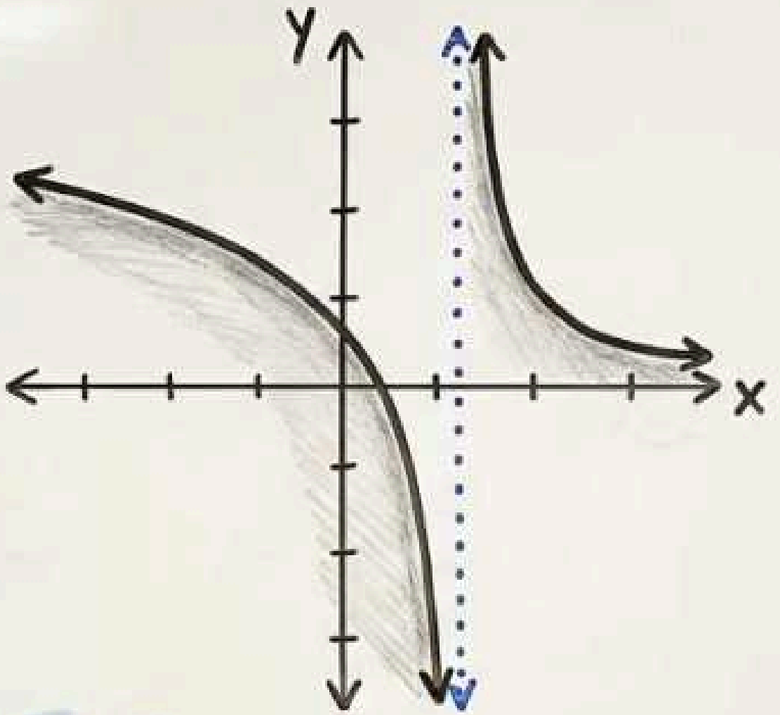
What is a Rational Function?

¶ 4.1

$$f(x) = \frac{P(x)}{Q(x)}$$

Polynomials



Used to describe ratios, averages, and rates.

Key Feature: Asymptotes — lines that the curve approaches more and more closely, a feature polynomial graphs lack.

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The Baseline: $f(x) = \frac{1}{x}$

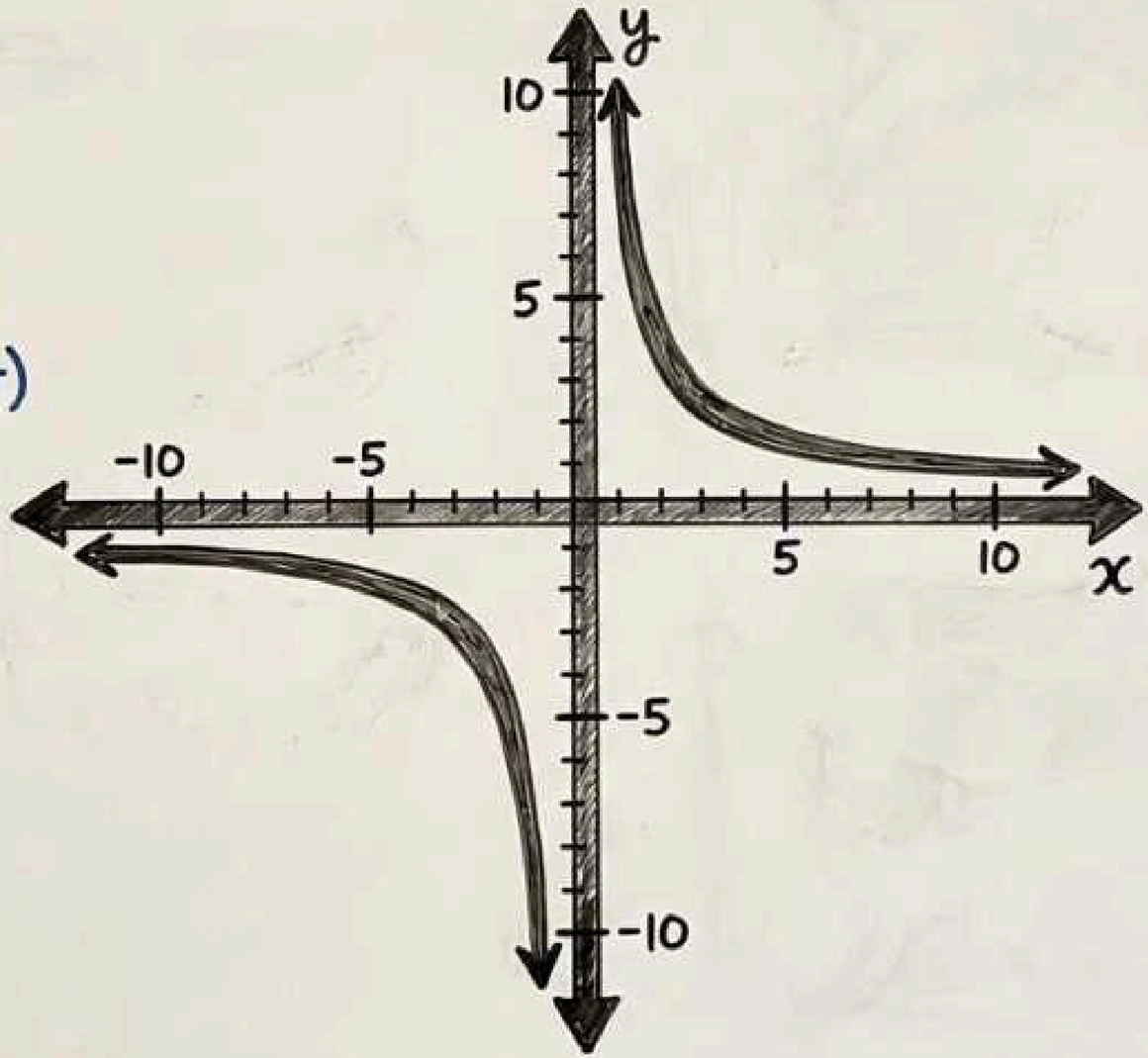
¶ 4.2

Domain: All real numbers except 0.

Large Inputs

$x = 1000 \rightarrow y = 0.001$

(Small Output)



Small Inputs

$x = 0.001 \rightarrow y = 1000$

(Huge Output)

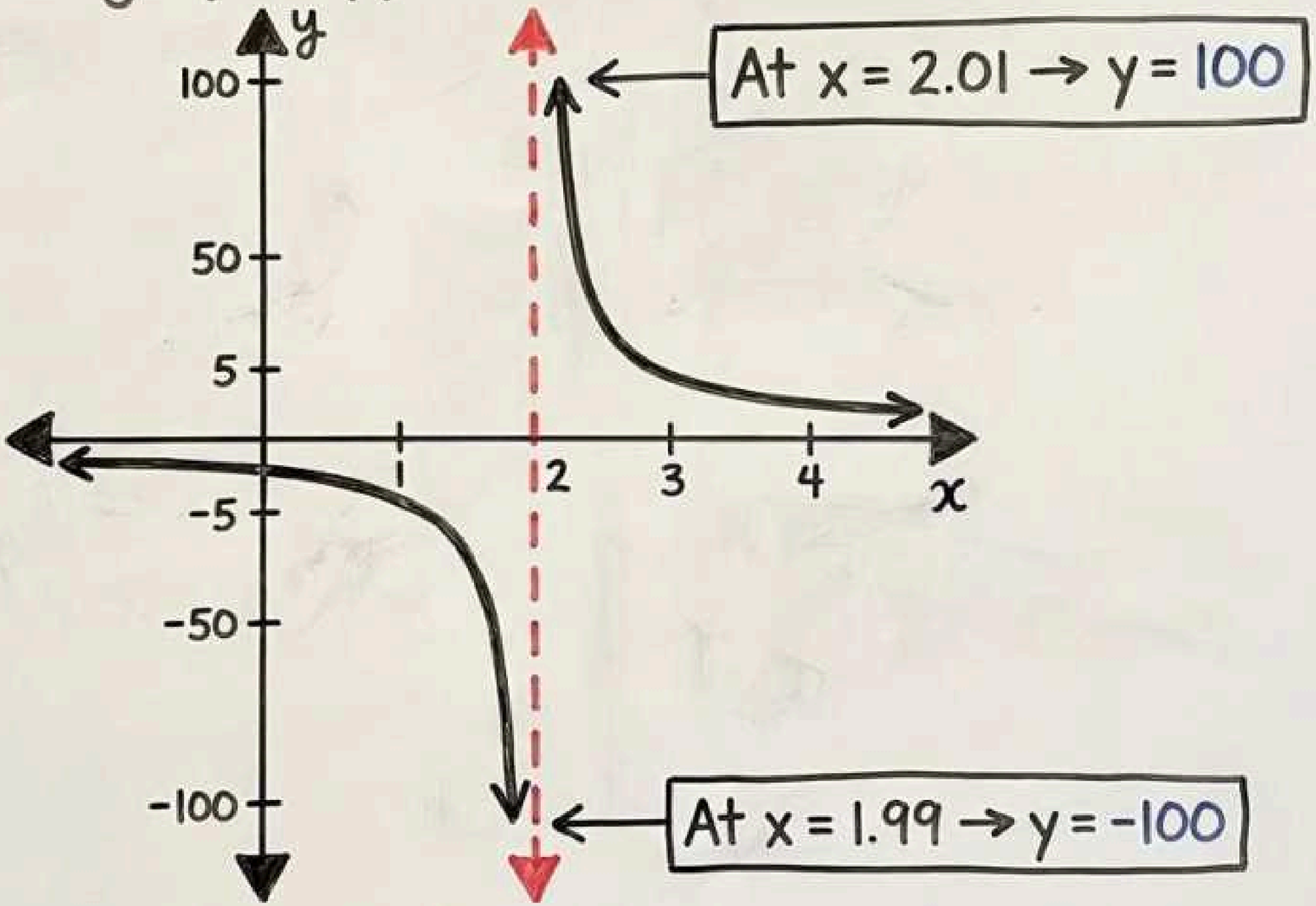
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Vertical Asymptotes

¶ 4.3

A vertical line the graph approaches but never reaches.

$$f(x) = \frac{1}{x-2}$$

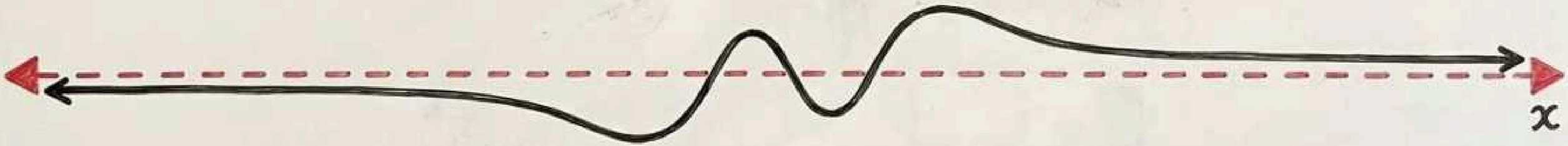


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Horizontal Asymptotes & End Behavior

¶ 4.4

Describes what happens as x gets very large (positive or negative). For $\frac{1}{x-2}$, the graph approaches 0.





Insight: Graphs can cross horizontal asymptotes in the middle, but settle toward them at the far ends.

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¶ 4.5

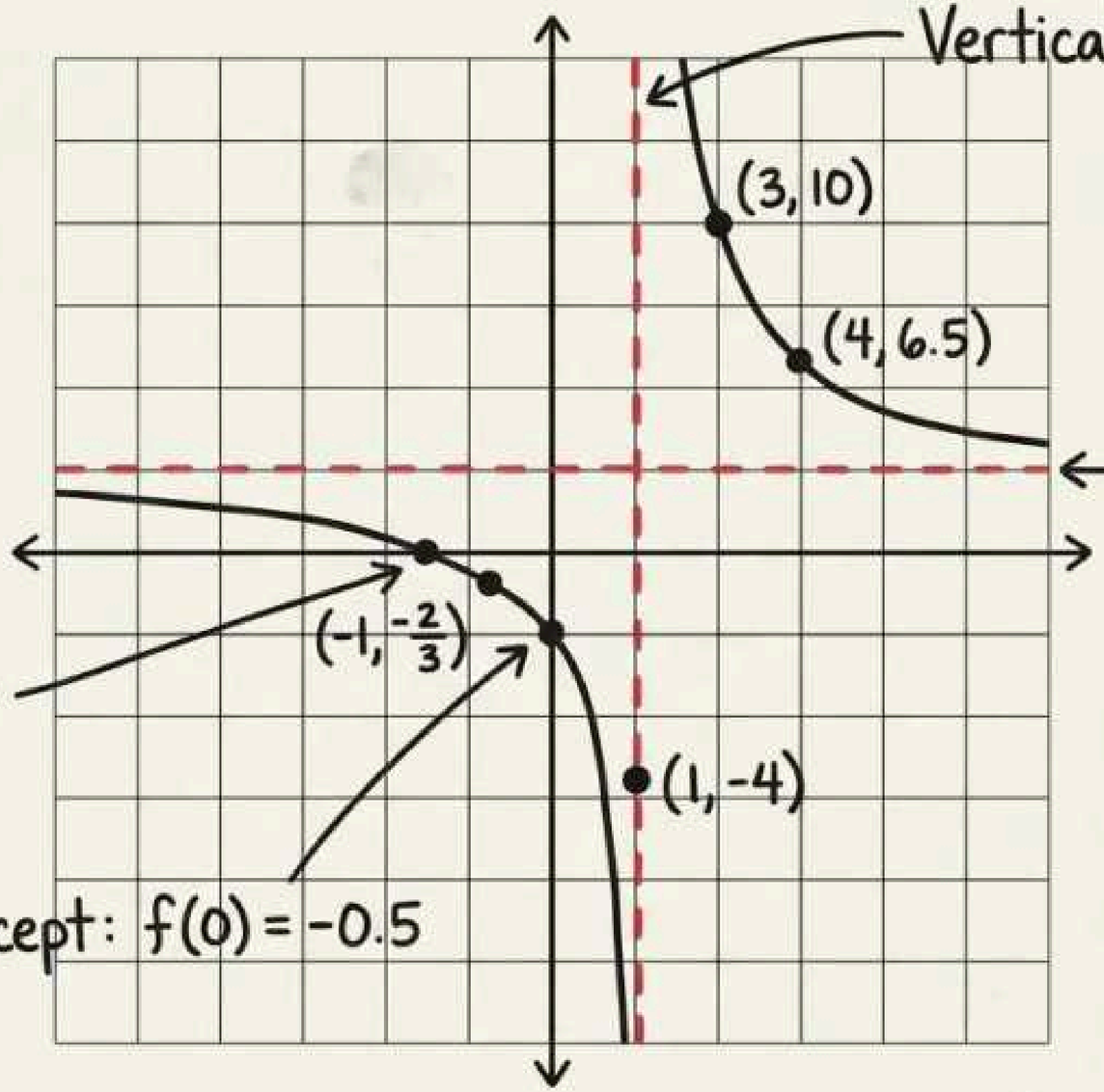
Rules for Horizontal Asymptotes

Numerator Degree < Denominator Degree	Degrees are Equal	Numerator Degree > Denominator Degree
↓ $y = 0$	↓ Ratio of leading coefficients. $\frac{3x+1}{x-2} \to y = 3$	↓ None

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Graphing Example: $f(x) = \frac{3x+1}{x-2}$

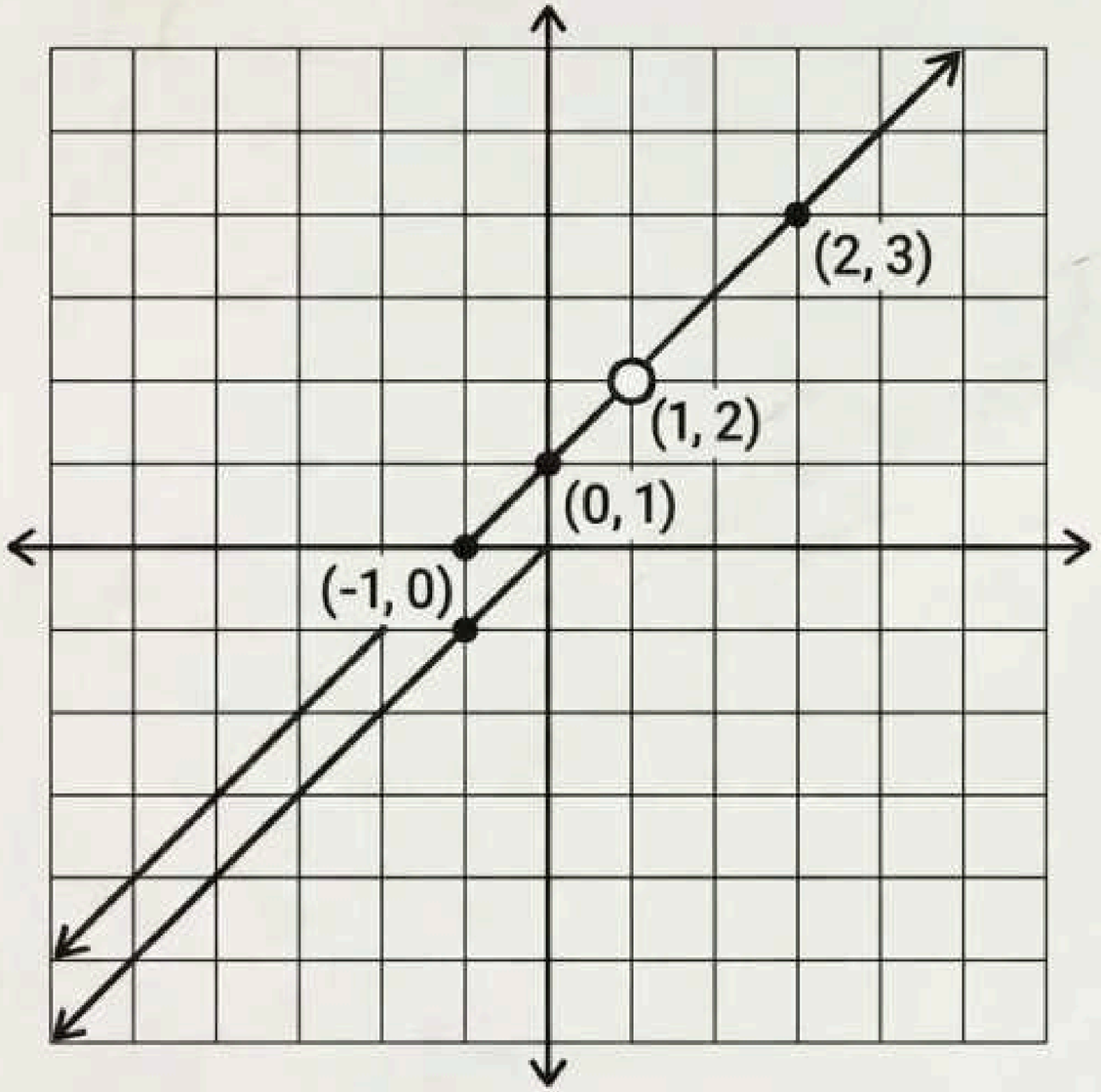
¶ 4.6



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Holes are Not Asymptotes

¶ 4.7



$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$= \frac{(x - 1)(x + 1)}{x - 1}$$

$$= x + 1$$

x=1 is still undefined, creating a single hole at (1, 2) instead of a vertical asymptote.

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¶ 4.8

The Golden Rule: Factor First, Then Cancel

Evaluate Every Factor

- Cancels completely → **Gives a Hole**
- Stays in Denominator → **Gives a Vertical Asymptote**
- Stays in Numerator → **Gives a Zero (x-intercept)**

Warning: Never decide on asymptotes before cancelling!

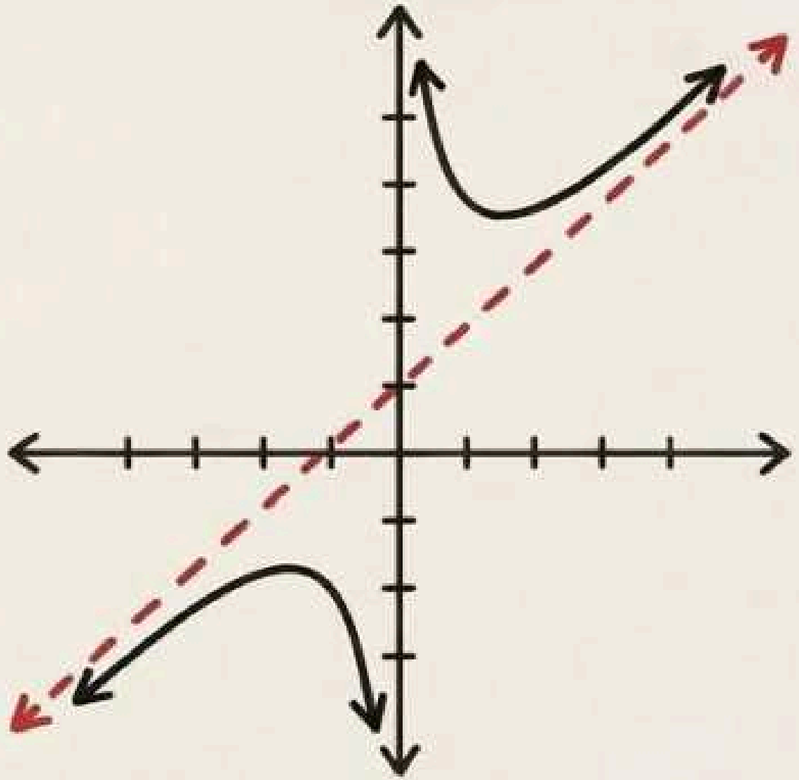
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Slant Asymptotes

¶ 4.9

Occurs when numerator degree is exactly one more than the denominator. Found via division.

$$
\begin{array}{r}
x+1 \\
x-1\overline{)\,x^2+0x+1} \\
-(x^2-x) \\
\hline
x+1 \\
-(x-1) \\
\hline
2
\end{array}
$$



$$f(x) = x + 1 + \frac{2}{x-1}$$

The quotient is x+1 (the asy).The remainder is $\frac{2}{x-1}$, which vanishes far from the origin.

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Master Checklist for Sketching

¶ 4.10

- [ ] 1. Factor completely and find domain.
- [ ] 2. Cancel common factors to locate holes.
- [ ] 3. Find x and y intercepts.
- [ ] 4. Find vertical asymptotes (from remaining denominator zeros).
- [ ] 5. Find horizontal or slant asymptote.
- [ ] 6. Test a point in each region to determine the sign.

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Application: Finding the Features

¶ 4.11

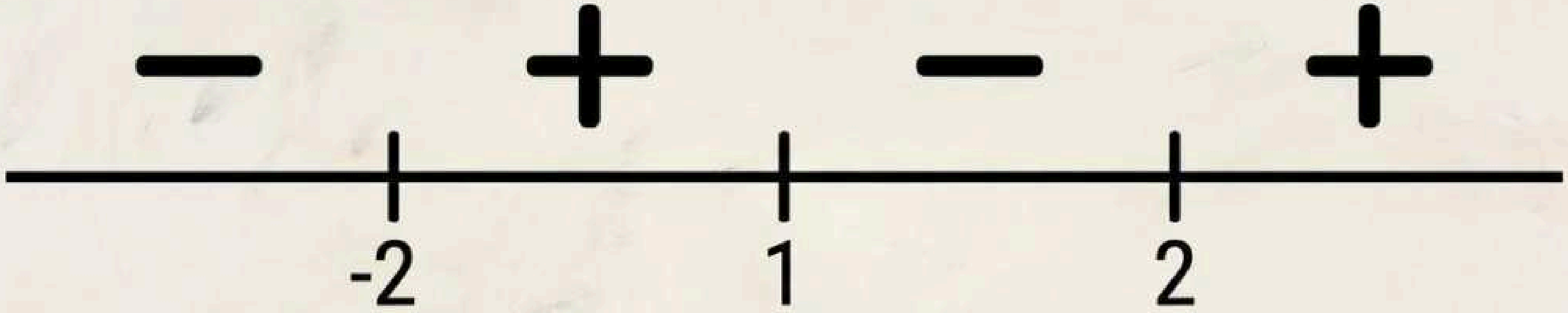
$$f(x) = \frac{x-1}{x^2-4}$$

1. Factor completely and find domain. → Denominator factors to (x-2)(x+2). No common factors to cancel in numerator. No holes.
2. Cancel common factors to locate holes.
3. Find x and y intercepts. → Intercepts: x-intercept = 1 (numerator zero), y-intercept = 0.25 (plug in x=0).
4. Find vertical asympto denominator zeros). → Vertical Asymptotes: x = 2 and x = -2 (denominator zeros).
5. Find horizontal or slant asymptote. → Horizontal Asymptote: y = 0 (Applying the Degree Rule: Numerator degree 1 < Denominator degree 2)

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Application: Testing the Regions

¶ 4.12



Test Points Data:

At $x = -3 \rightarrow y = -\frac{4}{5}$ (Negative / Below axis)

At $x = 0 \rightarrow y =$ positive (Above axis)

At $x = 1.5 \rightarrow y \approx -0.286$ (Negative / Below axis)

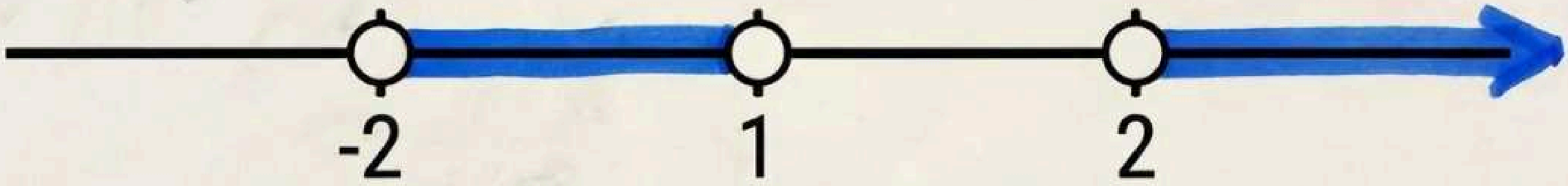
At $x = 3 \rightarrow y = \frac{2}{5}$ (Positive / Above axis)

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Solving Rational Inequalities

¶ 4.13

Solving $\frac{x-1}{x^2-4} > 0$.



Insight: Take the intervals where the function is positive. Endpoints are strictly excluded because the function is undefined at asymptotes (-2, 2) and exactly zero at the intercept (1).

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Solving Equations: Clear the Denominators

¶ 4.14

$$\frac{2}{x-1} = \frac{3}{x+2}$$

$$\Downarrow$$

$$2(x+2) = 3(x-1)$$

$$\Downarrow$$

$$2x + 4 = 3x - 3$$

$$\Downarrow$$

$$x = 7$$

Verification Check: ✓

Plug 7 back in. 2/6 = 1/3 and 3/9 = 1/3.

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The Danger of Extraneous Solutions

¶ 4.15

$$\frac{x^2}{x-3} = \frac{9}{x-3} \Rightarrow x^2 = 9 \Rightarrow x = 3, -3$$



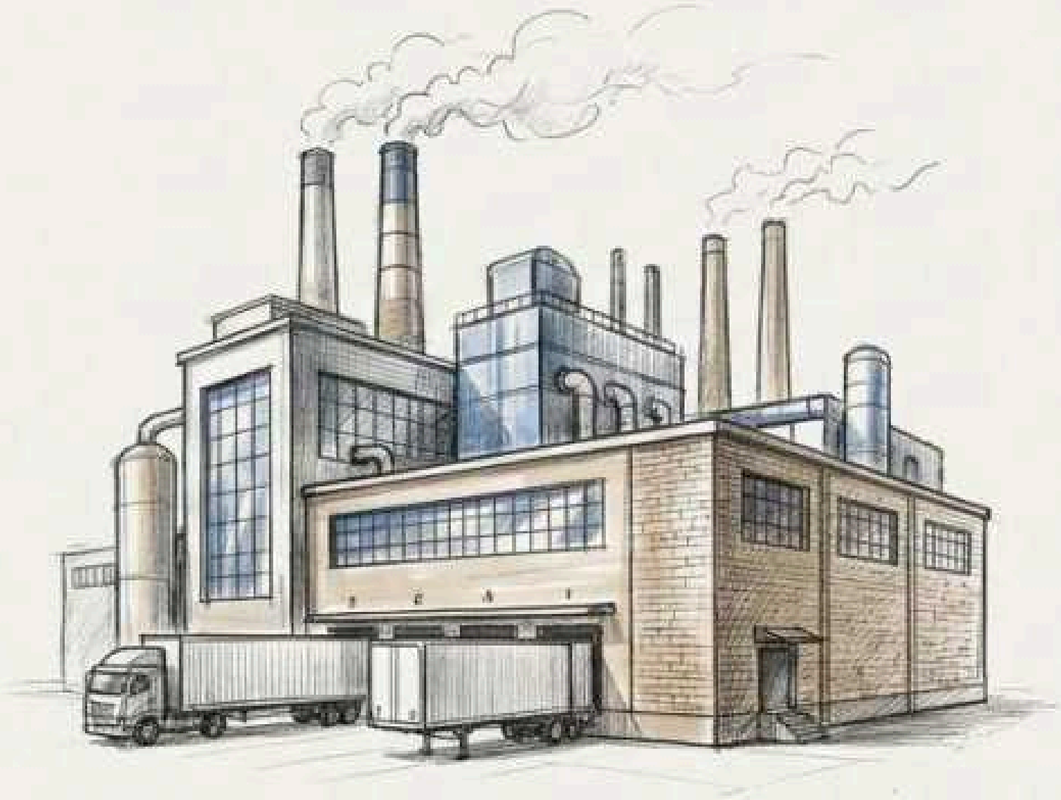
The value x=3 creates a zero in the denominator!
It is an extraneous (false) solution and must be rejected.
Only x=-3 survives.

Takeaway: Always check your answers in the original equation.

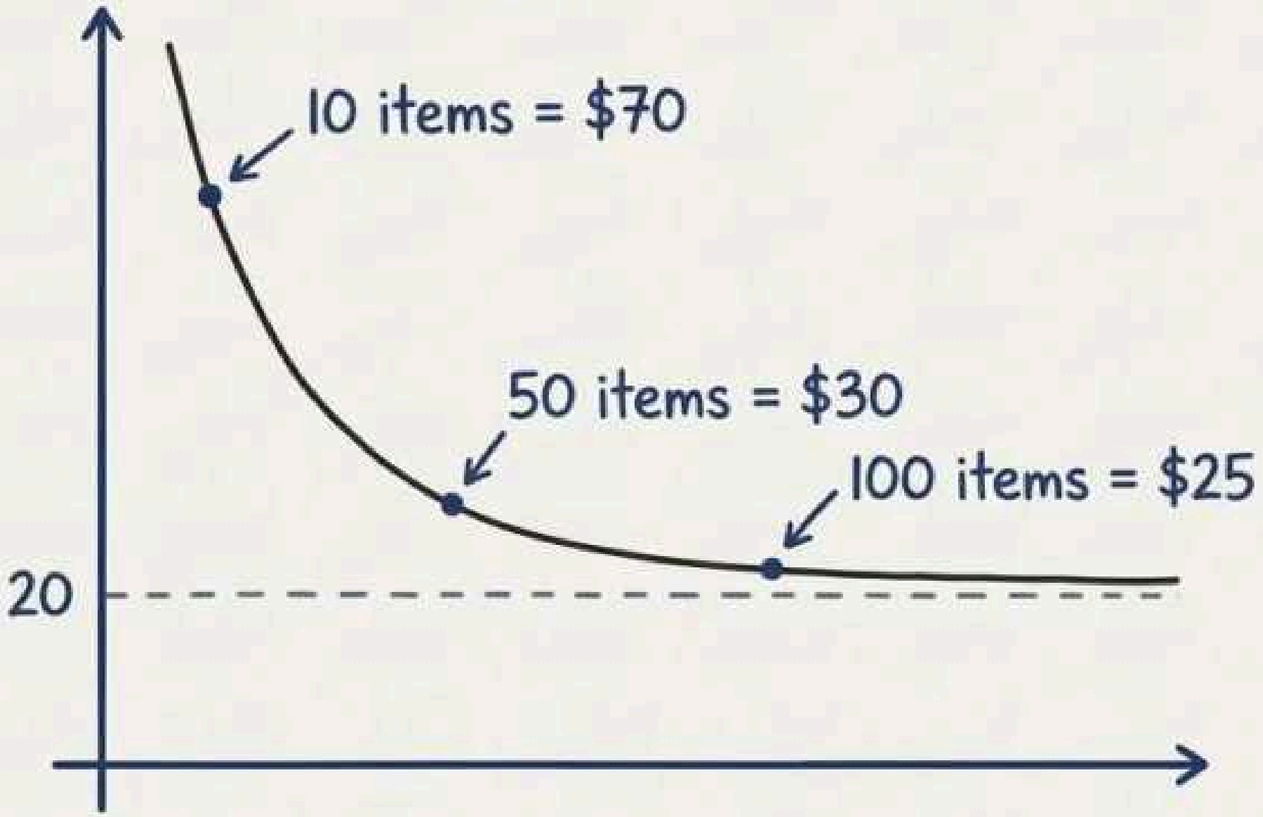
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Real-World Models: Average Cost

¶ 4.16



$$\text{Cost} = \frac{500 + 20x}{x} = 20 + \frac{500}{x}$$

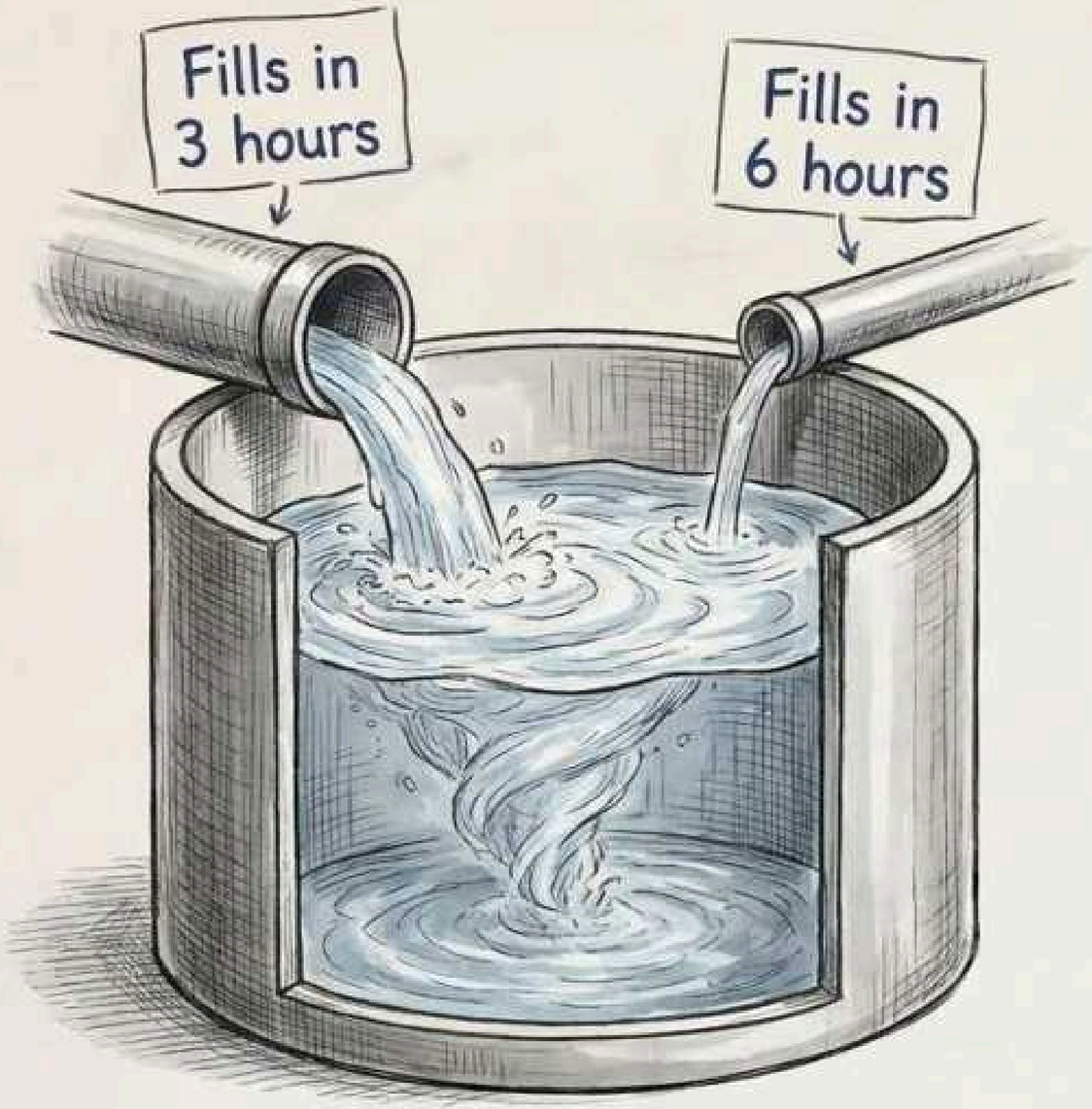


Insight: The horizontal asymptote (y = 20) acts as the absolute minimum floor of the cost per item, no matter how many are produced.

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Real-World Models: Combined Rates

¶ 4.17



$$\frac{1}{3} + \frac{1}{6} = \frac{1}{2} \text{ per hour.}$$

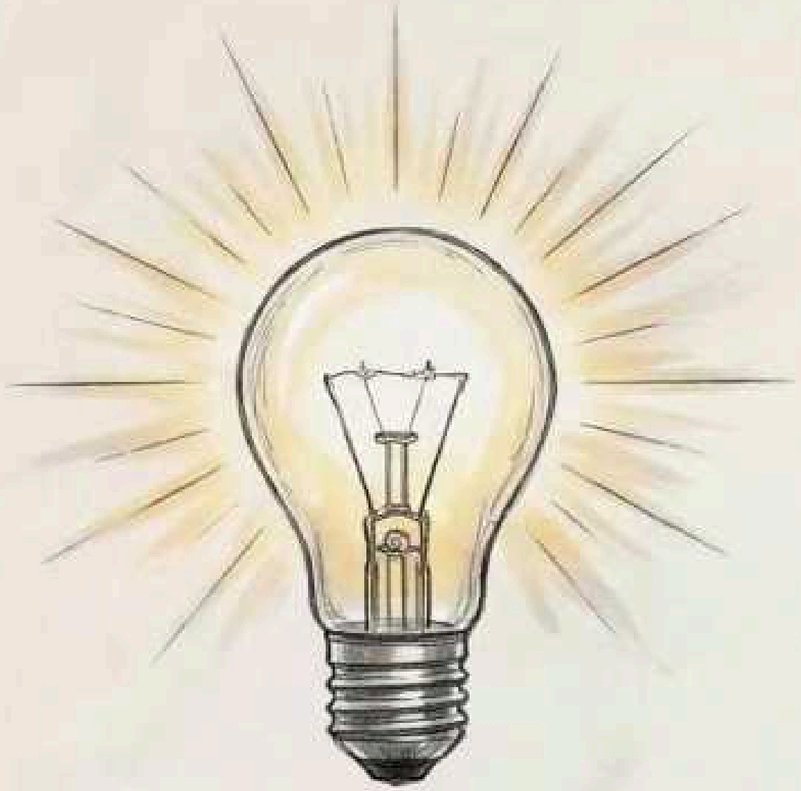
Total time t for both satisfies $\frac{1}{t} = \frac{1}{2}$, meaning $t = 2$ hours.

Key Insight:
Rates add; times do not!

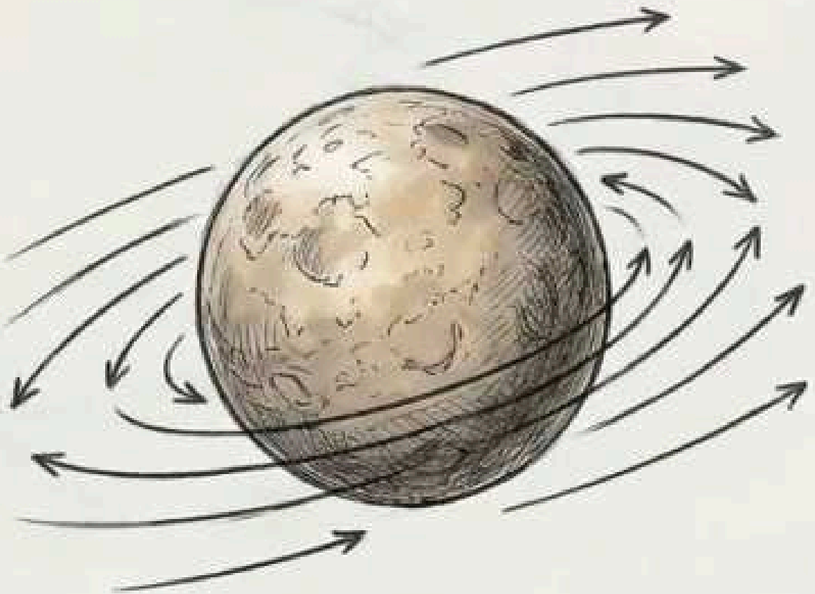
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Real-World Models: Inverse Variation

¶ 4.18



$$y = \frac{k}{x}$$



Example: If y=6 when x=4, then k=24. Therefore, when x=8, y=3.

Physics Insight: Light intensity and gravity vary inversely with the square of the distance ($y = \frac{k}{x^2}$).

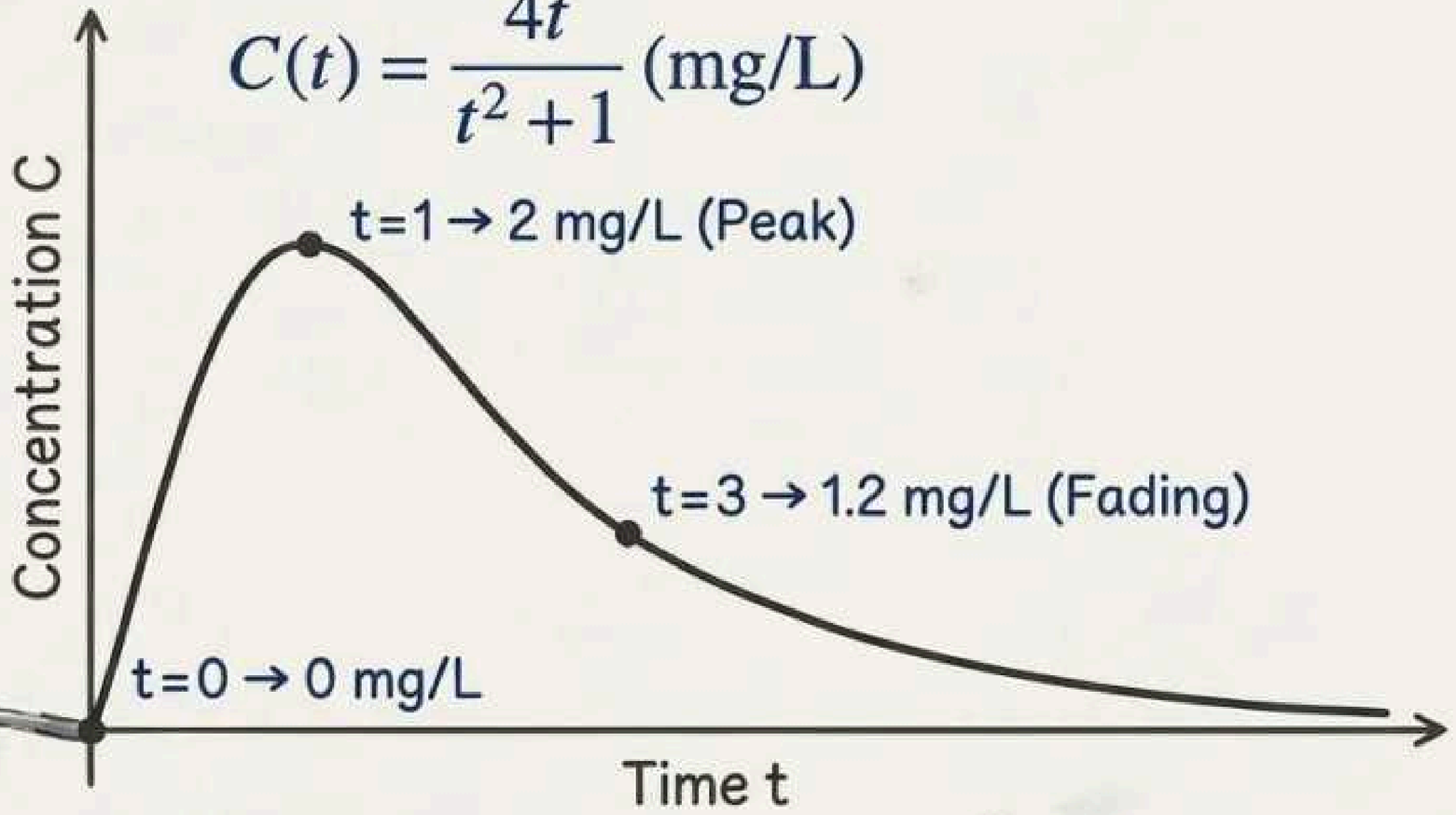
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Real-World Models: Drug Concentration

¶ 4.19



$$C(t) = \frac{4t}{t^2+1} \text{ (mg/L)}$$



Insight: The concentration rises, peaks, and slowly fades toward the horizontal asymptote (zero).

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¶ 4.20

Common Mistakes & Final Challenge

- Cancelling terms instead of factors $\frac{x+3}{x} \Rightarrow 1+3=4$
- Forgetting holes $\frac{x^2-4}{x-2} \Rightarrow x=2$ Hole!
- Calling every denominator zero an asymptote $\frac{x-2}{x-2} \Rightarrow$ Asymptote at $x=2$

Find all asymptotes and holes for:

$$f(x) = \frac{x^2 - 4}{x^2 - x - 2}$$

In Lecture 5, the variable moves into the exponent!

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