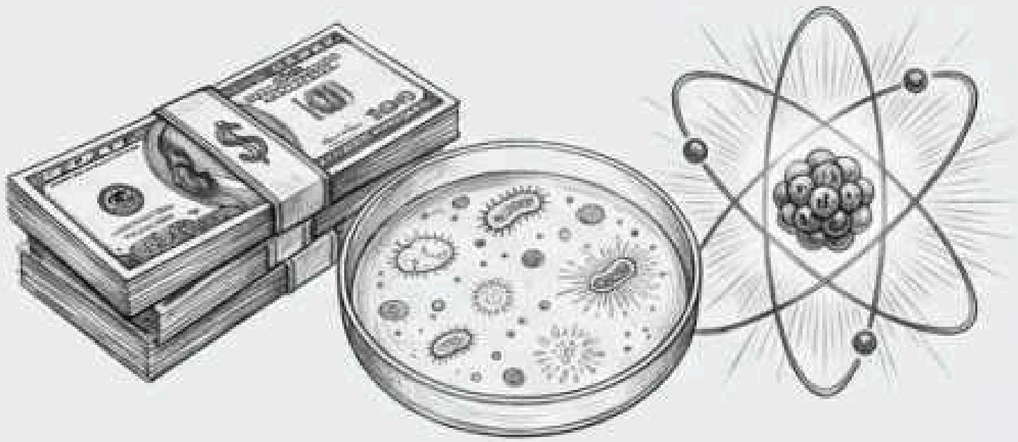
# Growth That Feeds on Itself: Exponentials & Logarithms

¶ 5.1

## The Engine: Exponential Functions



The variable is the exponent.

The Inverse

## The Control: Logarithms

Let us solve for the exponent

Tame quantities over huge scales.

**Exponentials build the scale. Logarithms decode it.**

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# Where Does the Variable Live?
## Power vs. Exponential

¶ 5.2

$f(x) = \boxed{x}^2$

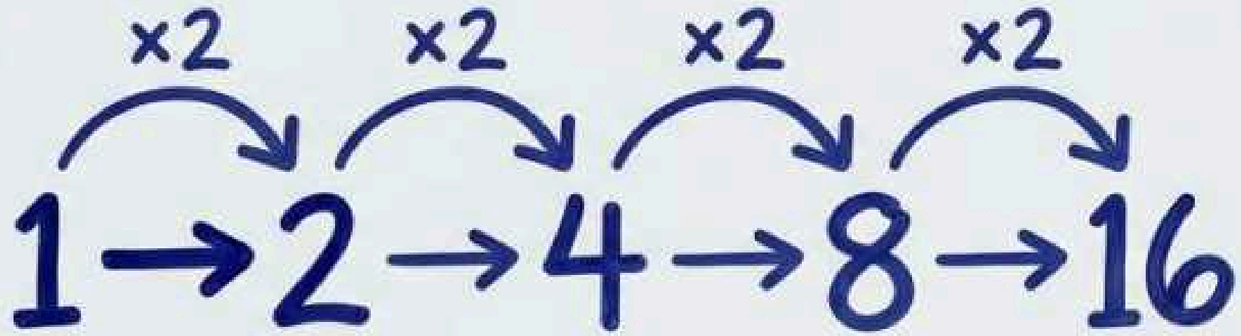
Variable is the BASE.

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$

$f(x) = 2^{\boxed{x}}$

Variable is the EXPONENT.

$(b > 0, b \neq 1)$




Each step multiplies by two.

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# The Exponent Rules: Know Them Cold

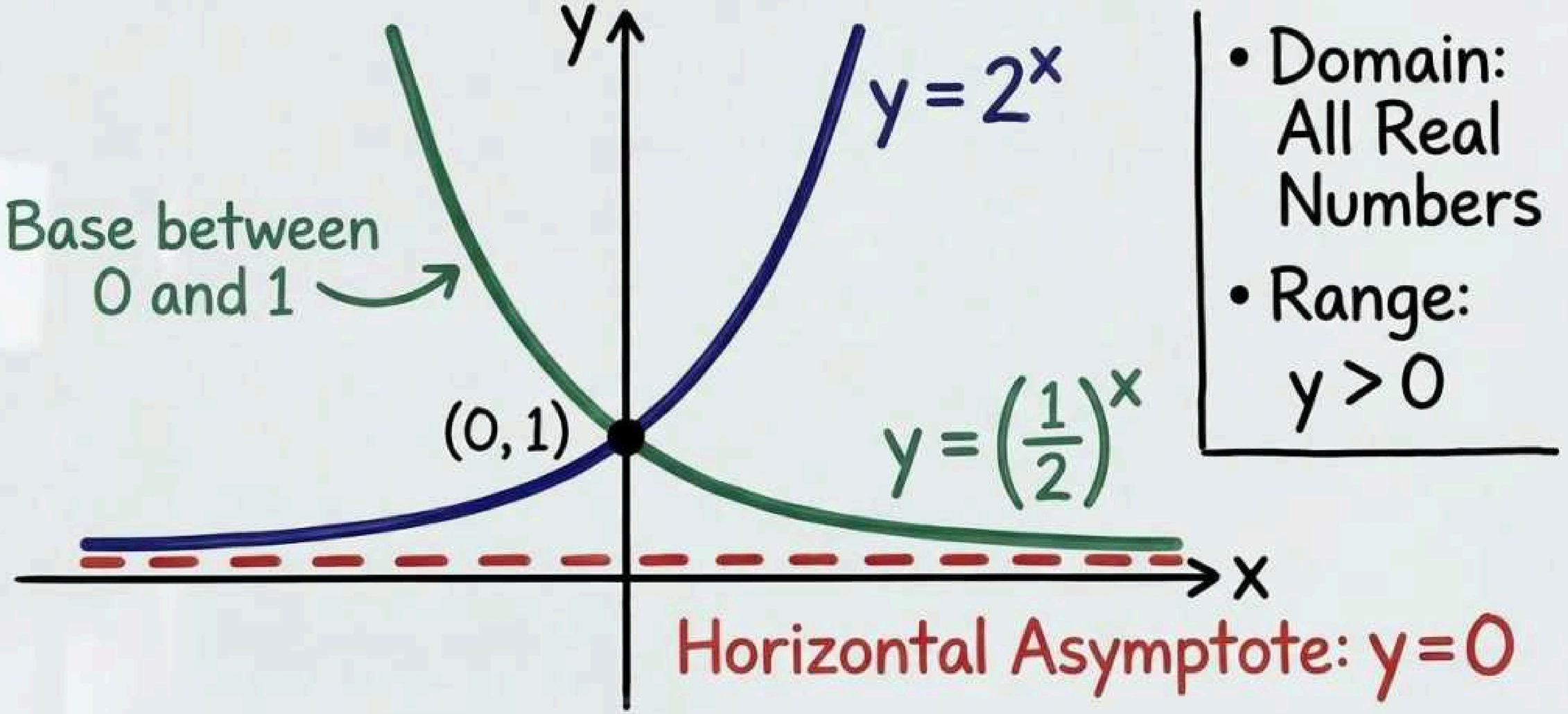
¶ 5.3

| | |
|---|---|
| $b^x \cdot b^y = b^{x+y}$ | Adding exponents |
| $(b^x)^y = b^{x \cdot y}$ | Multiplying exponents |
| $b^{-x} = \frac{1}{b^x}$ | Reciprocal $2^{-3} = \frac{1}{8}$ |
| $b^0 = 1$ | Yields one |
| $b^{1/2} = \sqrt{b}$ | Square root $9^{1/2} = 3$ |

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# Graphing Exponentials: Growth vs. Decay

¶ 5.4

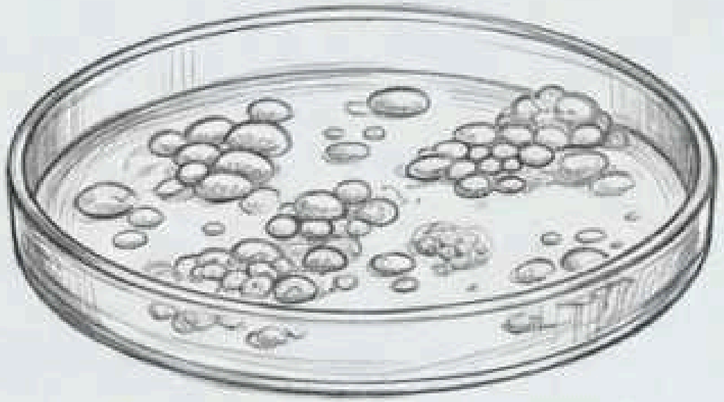



- Domain: All Real Numbers
- Range: $y > 0$

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# Exponential Population Models

¶ 5.5



$$\text{Amount} = (\text{Starting Amount}) \times b^{t}$$

500 cells double every 3 hours.

$$500 \times 2^{t/3}$$

t=9 hours

t=12 hours

$500 \times 2^{9/3} = 500 \times 2^{3} = 500 \times 8 = 4{,}000$ cells

$500 \times 2^{12/3} = 500 \times 2^{4} = 500 \times 16 = 8{,}000$ cells

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# The Classic Example: Compound Interest

¶ 5.6

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

P (Principal) r (Rate) n (Compounds per year) t (Years)

$1000 investment at 6% for 10 years.

| Annually | Monthly |
|---|---|
| n=1 | n=12 |
| $1,790.85 | $1,819.40 |

Compounding more often helps

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# The Discovery of e: The Limit of Compounding

¶ 5.7

...but only a little, and there is a limit.

$$\left(1+\frac{1}{n}\right)^n$$

$n = 1 \rightarrow 2$

$\downarrow$

$n = 12 \rightarrow 2.613$

$\downarrow$

$n \rightarrow \infty \rightarrow \boxed{2.71828\ldots}$

This limit is the number **e**,
the natural base of growth.

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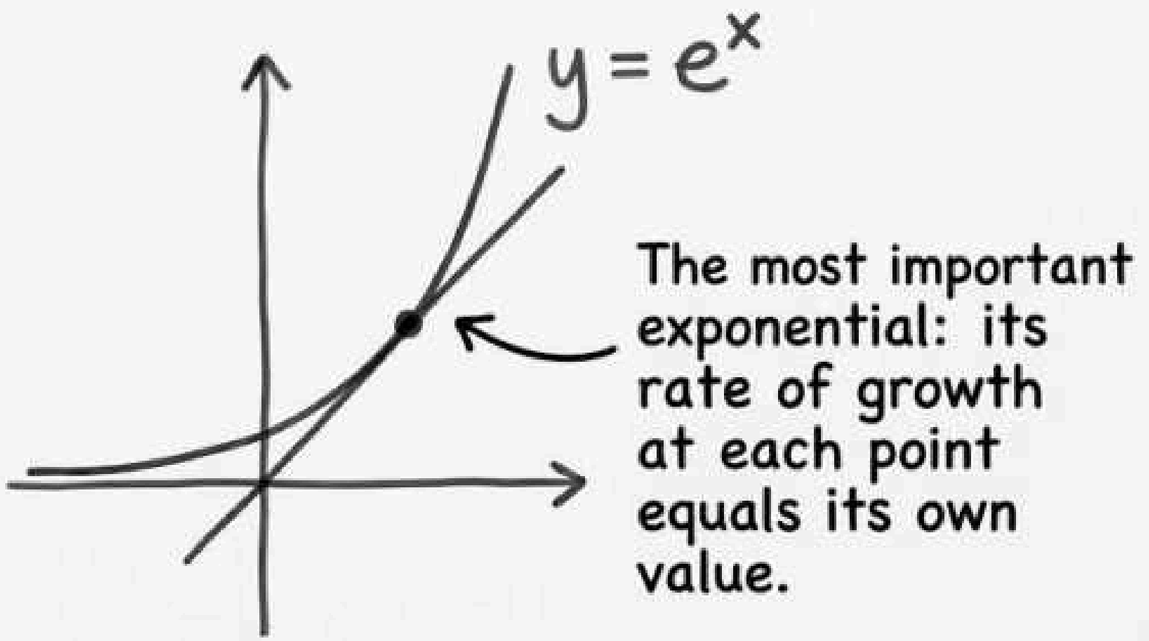
¶ 5.8

# Continuous Compounding

$$A = P \cdot e^{rt}$$

$1000 at 6% for 10 years

Amount = $1,822.12 ✓ Maximum possible yield




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# Introduction to Logarithms: "What exponent is needed?"

¶ 5.9

$$\log_b(y) = x \iff b^x = y$$

## Cyclic Translation Model

$\log_2(8) = 3$

$2^{\text{what}} = 8?$

$\log_{10}(1000) = 3$

$10^{\text{what}} = 1000?$

$\log_5\left(\frac{1}{25}\right) = -2$

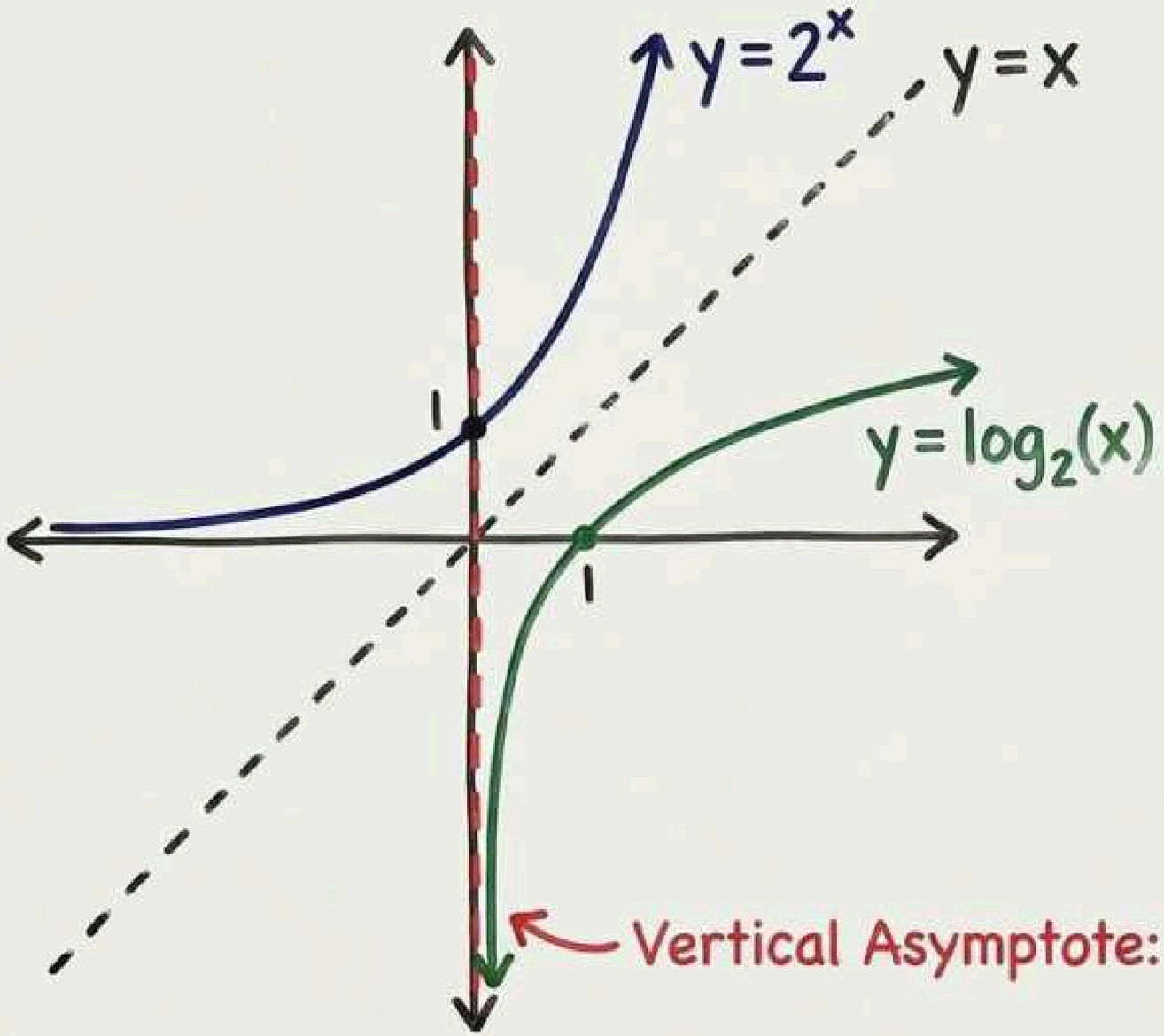
$5^{\text{what}} = \frac{1}{25}?$

**A logarithm IS an exponent.**

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# Graphing Logarithms as Inverses

¶ 5.10




Domain: $x > 0$
(No power of a positive base is zero or negative).

Range: All Real Numbers.

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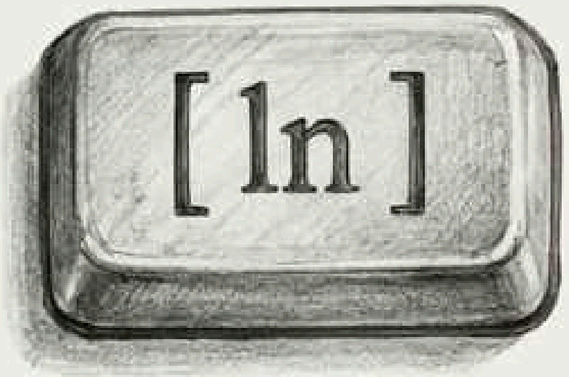
# Special Bases: Common and Natural Logs

¶ 5.11




Uses Base 10

$$\log(10) = 1$$




Uses Base e

$$\ln(e) = 1$$

$$\ln(1) = 0$$

Both have buttons on every calculator.
They follow the same fundamental logic.

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# The Laws of Logarithms

¶ 5.12

Logarithms turn multiplication into addition.

$$\log(x \cdot y) = \log(x) + \log(y)$$

$$\log_2(32) = \log_2(8 \cdot 4) = 3 + 2 = 5$$

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

$$\log(x^p) = p \cdot \log(x)$$




$$\log(10 + 90) = \log(100) = 2$$

$$\log(10) + \log(90) = 1 + 1.954 = 2.954$$

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# The Change of Base Formula

¶ 5.13

To change bases, divide logs.

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

---

Calculate $\log_2(50)$.

(Calculator only has ln and log)

Step 1: Set up the quotient → $\frac{\ln(50)}{\ln(2)}$

Step 2: Evaluate numer/denom → $\frac{3.912}{0.693}$

Step 3: Final division → ≈ 5.644

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# Solving Exponential Equations (Part 1)

¶ 5.14

## Logarithms

$$2^x = 50$$

⇩ Apply logarithm

$$x = \log_2(50)$$

⇩

$$x \approx 5.644$$

## Matching Bases

$$3^{2x-1} = 81$$

⇩ Note: $81 = 3^4$

$$3^{2x-1} = 3^4$$

⇩ Equate exponents

$$2x - 1 = 4$$

⇩

$$x = 2.5$$

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# Solving Base-e Equations

¶ 5.15

$$5e^{0.3t} = 40$$

↓ Divide by 5

$$e^{0.3t} = 8$$

↓ Take natural log (ln) of both sides

$$0.3t = \ln(8)$$

↓ Evaluate $\ln(8) \approx 2.079$

$$0.3t \approx 2.079$$

↓ Divide by 0.3

$$\mathbf{t \approx 6.931}$$

✓ Substitute back to check: $5e^{0.3(6.931)} \approx 40$. It works.

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# Solving Logarithmic Equations

¶ 5.16

Logarithmic equations are solved the other way: by exponentiating.

$$\log_2(x+3) = 4$$

$$\downarrow$$

$$2^4 = x + 3$$

$$\downarrow$$

$$16 = x + 3$$

$$\downarrow$$

$$x = 13$$

Always check the result in the original equation.
Logarithms accept only positive inputs.

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# When Checking Matters: Extraneous Solutions

¶ 5.17

$$\log(x) + \log(x-3) = 1$$

↓ Combine:

$$\log(x(x-3)) = 1$$

↓ Exponentiate:

$$x^2 - 3x = 10^1 = 10$$

↓ Solve Quadratic:

$$x^2 - 3x - 10 = 0 \rightarrow x = 5 \text{ or } x = -2$$

$x = -2$

Plug into $\log(-2)$

Rejected. Inputs cannot be negative.

$x = 5$ ✓

Plug into $\log(5) + \log(2) = \log(10) = 1$

Valid Solution.

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¶ 5.18

# Application: Doubling Time & The Rule of 72

How long until money doubles at 5% compounded annually?

## (The Exact Math)

Solve: $1.05^t = 2$

Apply logs: $t = \frac{\ln(2)}{\ln(1.05)}$

Result: $\approx 14.2$ years



## (The Estimate)

The Rule of 72

$$\frac{72}{\text{Interest Rate}}$$

$$\frac{72}{5} = 14.4 \text{ years}$$

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# Application: Radioactive Half-Life

¶ 5.19

Carbon-14 has a half-life of 5,730 years.

$$\text{Fraction remaining} = \left(\frac{1}{2}\right)^{t/5730}$$

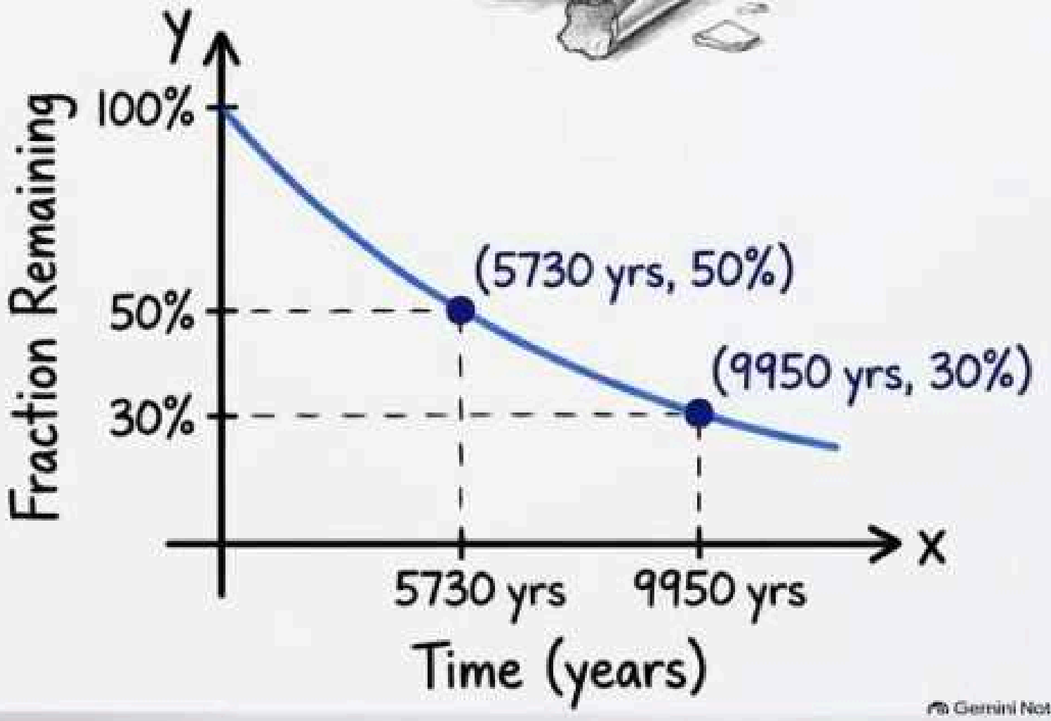


A sample with 30% remaining.

$$0.3 = \left(\frac{1}{2}\right)^{t/5730}$$

↓ Solve for t using logs...

$$t \approx 9{,}950 \text{ years}$$

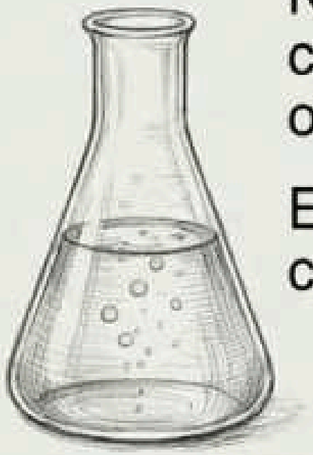



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# Logarithmic Scales: Measuring by Exponent

¶ 5.20

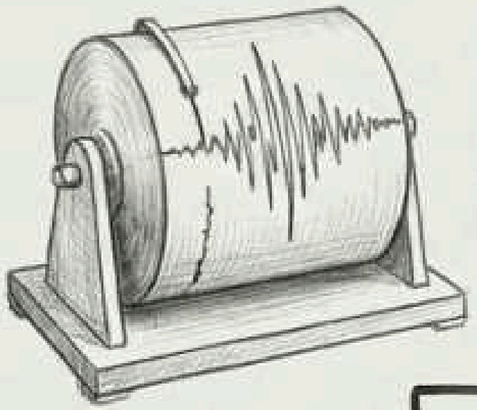
## pH



Negative common log of hydrogen.

Each step = 10x concentration.

## Earthquakes



Magnitude step = 10x ground motion.

## Decibels



10 dB step = 10x sound power.

$10^1$ $10^2$ $10^3$ ...

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# Closing Problem & Looking Ahead

¶ 5.21

A culture of 200 cells grows by a factor of 3 every 4 hours. When does it reach 10,000?

Write the Model: $\text{Amount} = 200 \cdot 3^{t/4}$

Set the Target: $10{,}000 = 200 \cdot 3^{t/4}$

Solve with Logs: $50 = 3^{t/4} \rightarrow \ln(50) = \frac{t}{4} \cdot \ln(3) \rightarrow t \approx 14.25$ hours

---

**Next Lecture**

Lecture 6 looks ahead to calculus, exploring sequences, series, and limits. →

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