The Doorway to Calculus

Sequences & Series

Algebraic Roots: Lists and sums of numbers

Limits

The Bridge: The idea separating calculus from everything before it

Calculus

The Destinations: The tangent line problem & the area problem

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Definitions: Sequence vs. Series

¶ 6.2

Sequence

Definition: An ordered list of numbers (terms).

Function: Inputs are counting numbers (1, 2, 3...) where the nth term denotes position.

Example: 3, 7, 11, 15...

Series

Definition: The sum of the terms of a sequence.

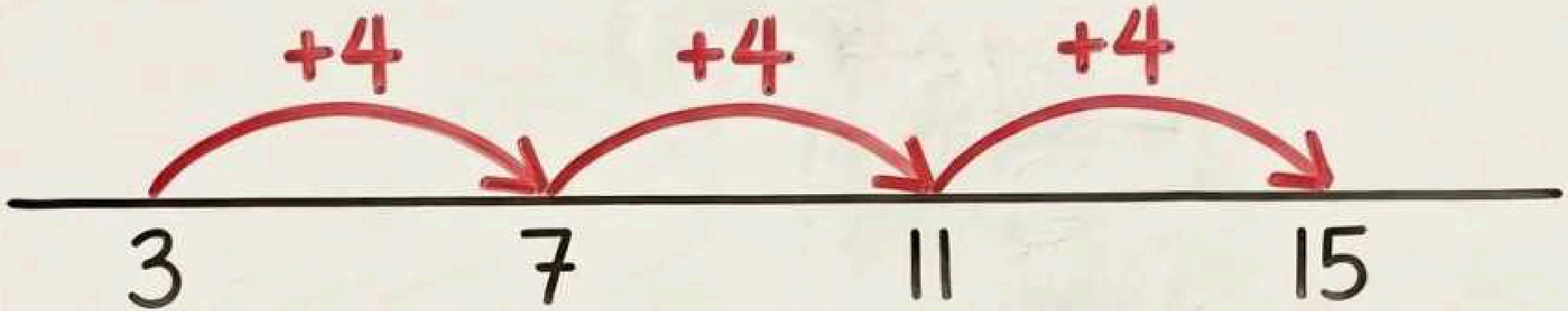
Example: 3 + 7 + 11 + 15...

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Arithmetic Sequences

¶ 6.3

Each term differs from the last by a constant (d, the common difference).



$$a_n = a_1 + (n-1)d$$

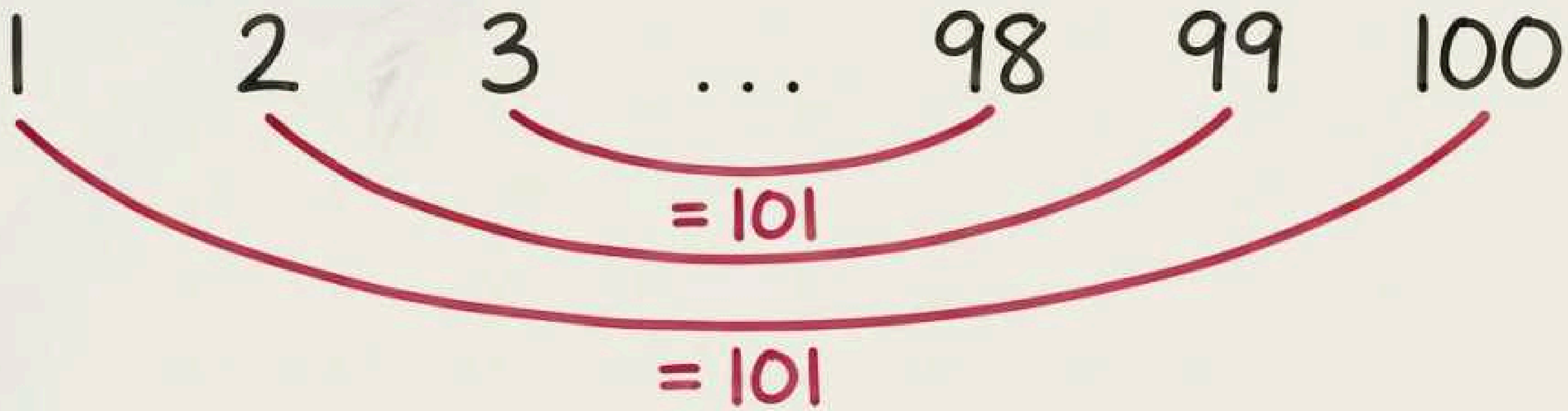
Finding the 20th term:

$$3 + (19 \times 4) = 79$$

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Arithmetic Series & Gauss's Trick

$$S_n = n \times \left(\frac{a_1 + a_n}{2} \right)$$



$$100 \times \left(\frac{101}{2} \right) = 5{,}050$$

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Applying the Pairing Trick

Dataset: Summing 20 terms from 3 up to 79.

Method 1: Formula:

$$\text{Total} = 20 \times \left(\frac{3 + 79}{2}\right)$$

$$\text{Total} = 20 \times 41 = \boxed{820}$$

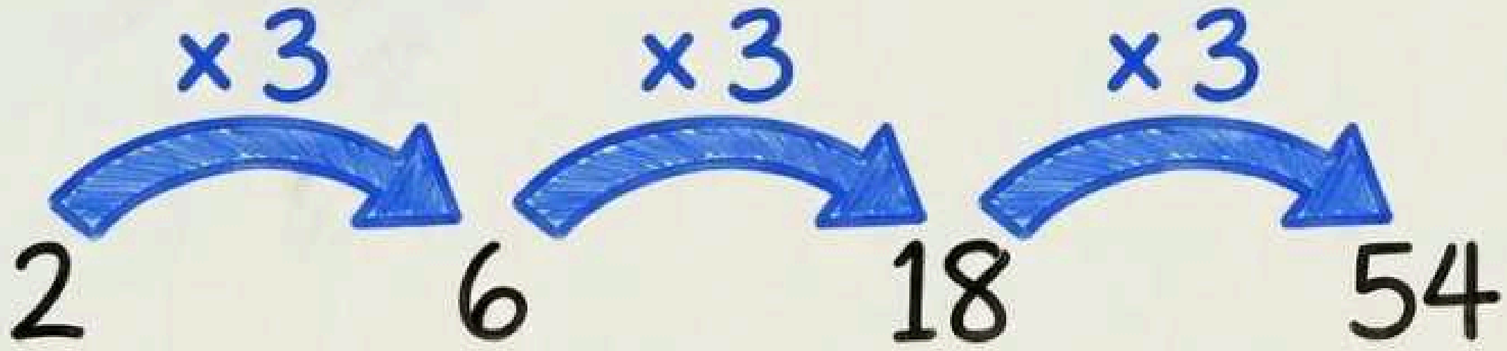
Method 2: Check by Pairing:

10 pairs x 82 (sum of each pair) = $\boxed{820}$

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Geometric Sequences

Each term is a constant multiple of the last (r, the common ratio).



$$a_n = a_1 \times r^{n-1}$$

The 8th term: $2 \times 3^7 = 2 \times 2{,}187 = 4{,}374$

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Geometric Series

$$S_n = a_1 \times \left(\frac{1-r^n}{1-r}\right)$$

Verification Table

Summing: 2, 6, 18, 54 (n=4, r=3)

By Formula:

$$2 \times \left(\frac{1-3^4}{1-3}\right) \rightarrow 2 \times \left(\frac{1-81}{1-3}\right)$$

$$\rightarrow 2 \times \left(\frac{-80}{-2}\right) = 80$$

By Addition:

$$2+6+18+54$$

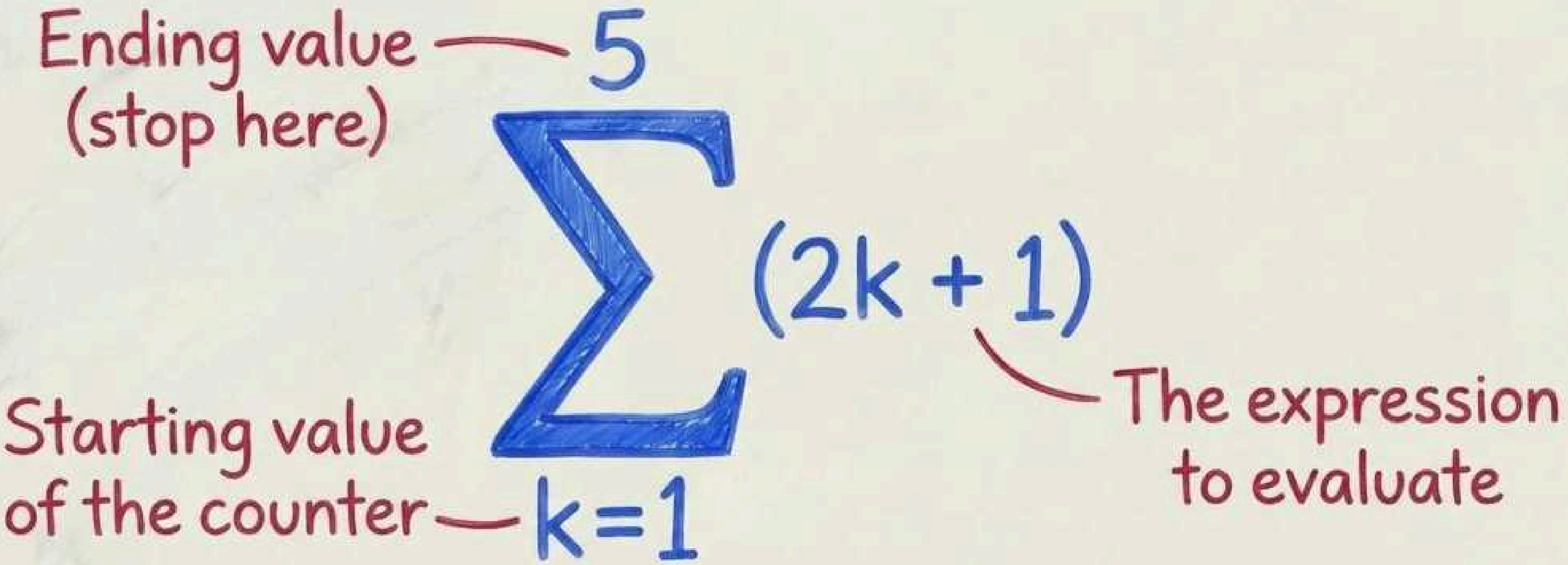
$$= 80$$



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Sigma Notation (Shorthand for Sums)



Summing from k=1 to 5: $3+5+7+9+11=35$

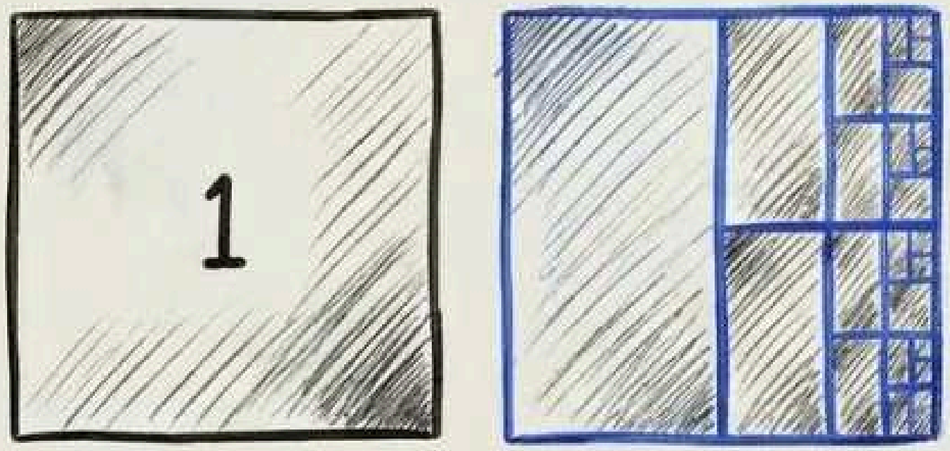
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Infinite Geometric Series: Convergence

A finite sum is possible if the ratio is between -1 and 1. Sum approaches $\frac{a_1}{1-r}$.

Left Panel (Halves)

$1 + 0.5 + 0.25\ldots = 2$



Right Panel (Repeating Decimals)

$0.333\ldots$ is calculated as:

$$\frac{\frac{3}{10}}{1-\left(\frac{1}{10}\right)} = \frac{1}{3}$$

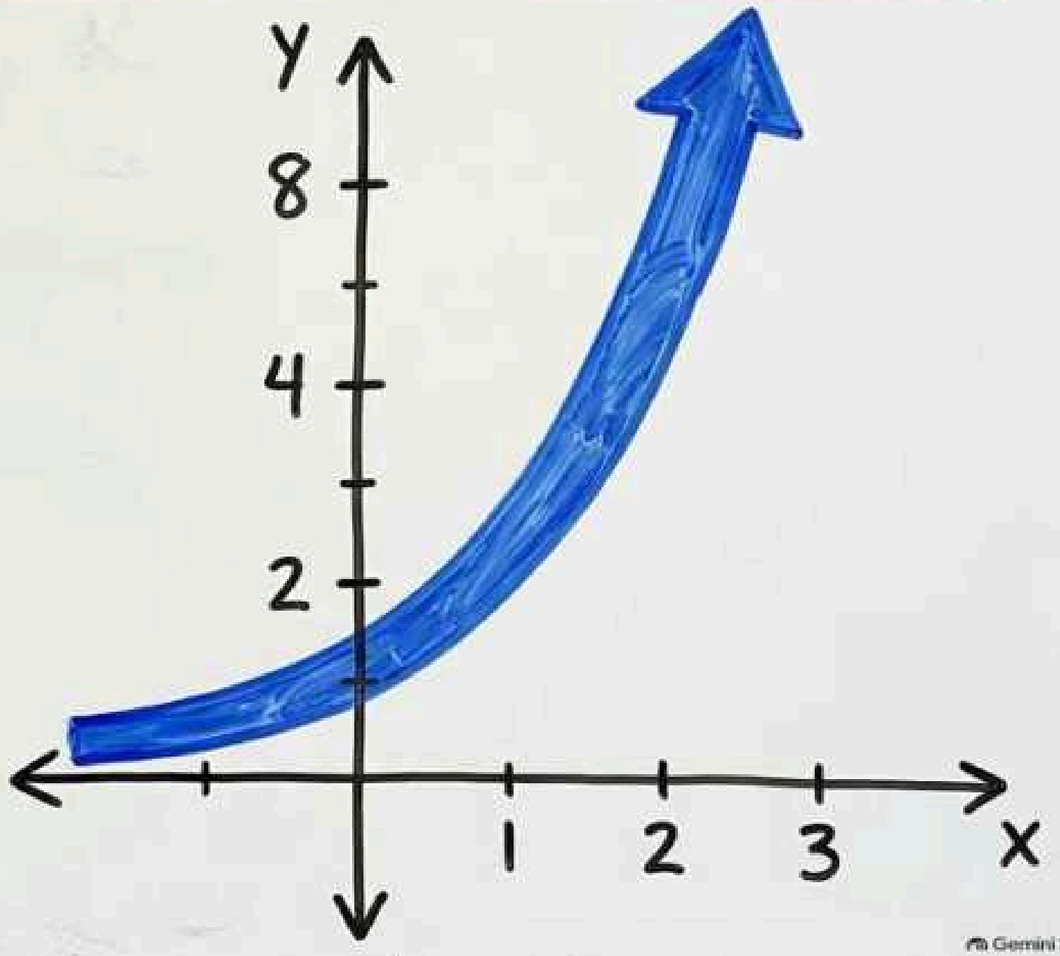
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Infinite Geometric Series: Divergence

If $|r| \geq 1$, terms do <u>not</u> shrink. The series diverges.

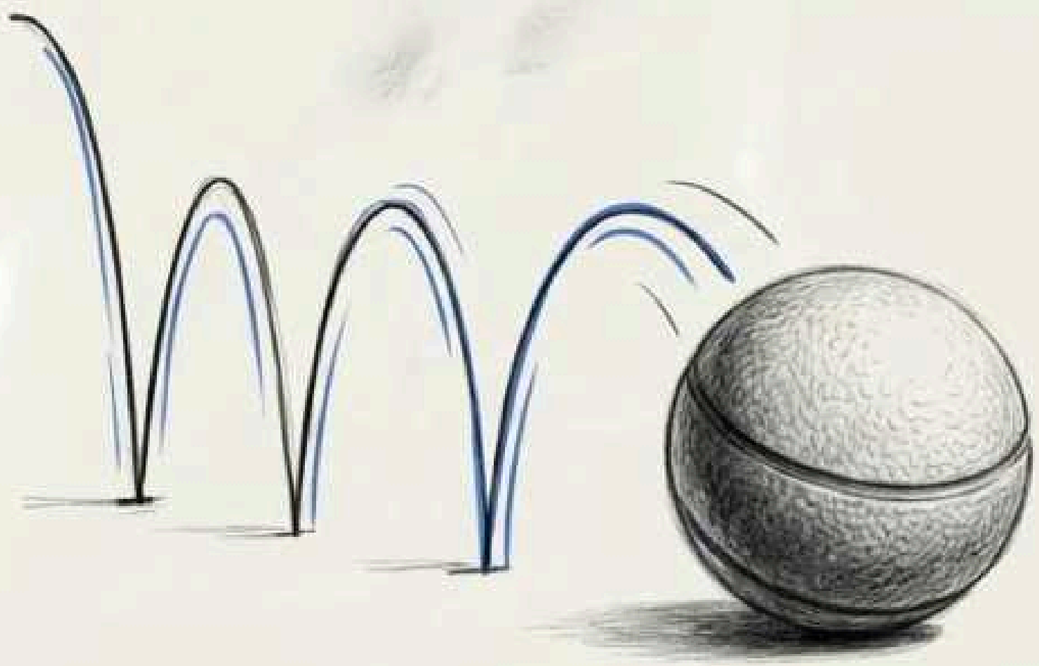
Example: $1 + 2 + 4 + 8 \ldots$
(grows without bound)

Convergence requires terms to shrink toward zero fast enough. This test (ratio < 1 in size) is our first example of the limit of a sum.



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Application: The Bouncing Ball



Setup: Dropped from 10 ft, bounces back to 60% height (r=0.6).

The Math:

First drop: 10 ft

Up-and-down bounces: $2 \times \left(\frac{6}{1-0.6} \right) = 30$ ft

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Total Path: $10 + 30 = 40$ ft

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Application: Savings Plans



Setup: Deposit $100/month into an account earning 0.5% interest/month.

The Math (Finite Geometric Sum):

$$\text{12 months balance} = 100 \times \left(\frac{1.005^{12} - 1}{0.005} \right)$$

Result: approx **$1,233.56**

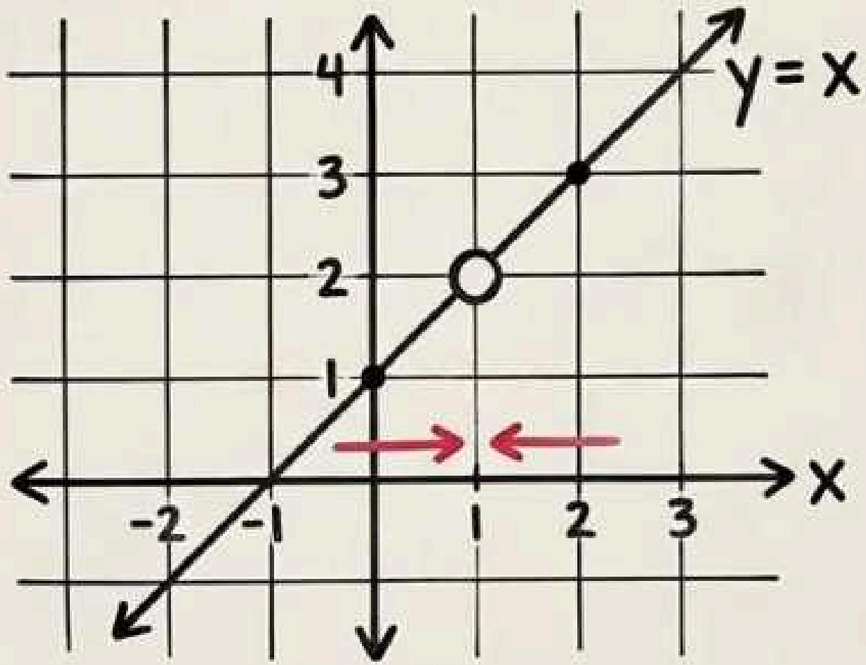
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The Concept of Limits

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The limit of $f(x)$ as x approaches c is L if values get as close as we like to L without x equaling c.

$$f(x) = \frac{x^2-1}{x-1} \quad \text{(Undefined at } x=1\text{)}$$



$x = 0.90 \rightarrow$	$f(x) = 1.90$
$x = 0.99 \rightarrow$	$f(x) = 1.99$
$x = 1.01 \rightarrow$	$f(x) = 2.01$

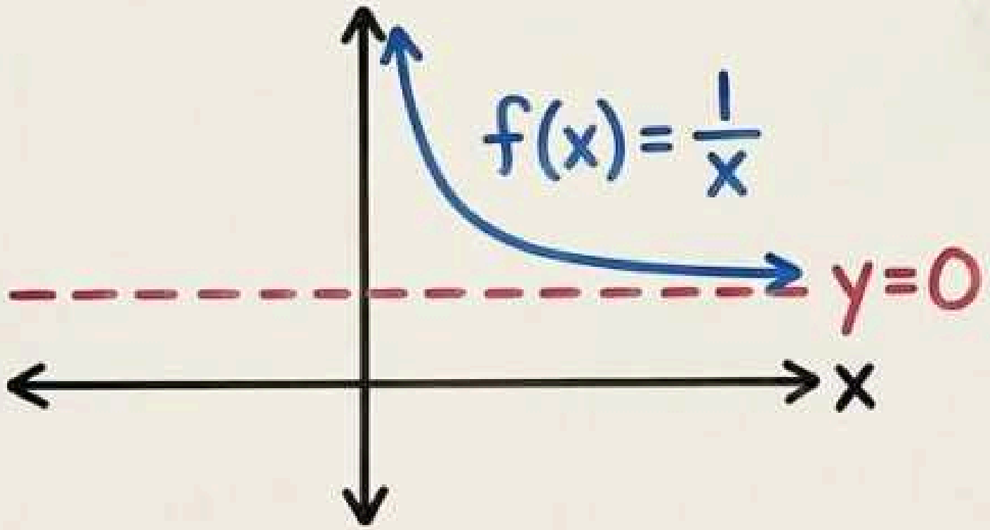
Conclusion: The limit is 2.

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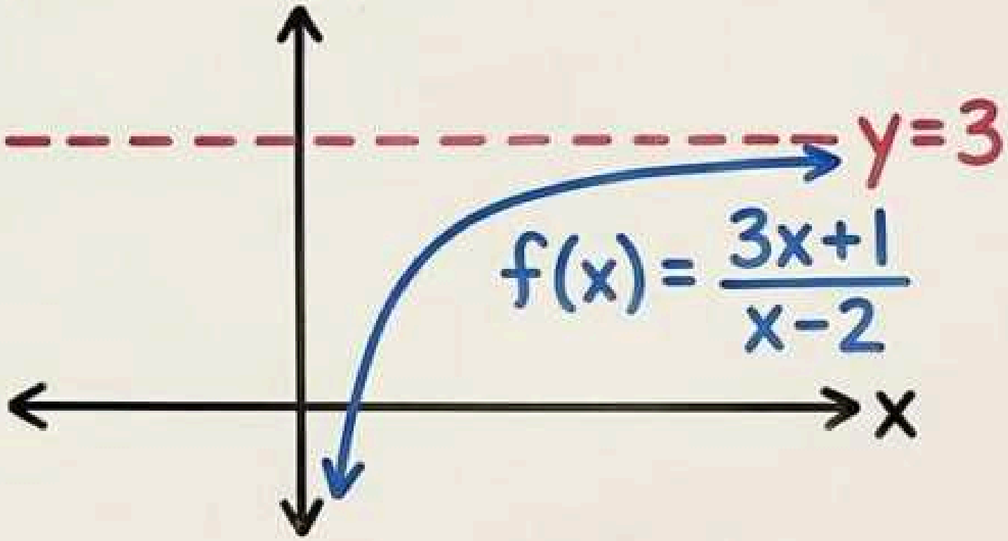
Limits at Infinity (End Behavior)

A horizontal asymptote is simply a limit at infinity.

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$$



$$\lim_{x\to\infty}\frac{3x+1}{x-2}=3$$



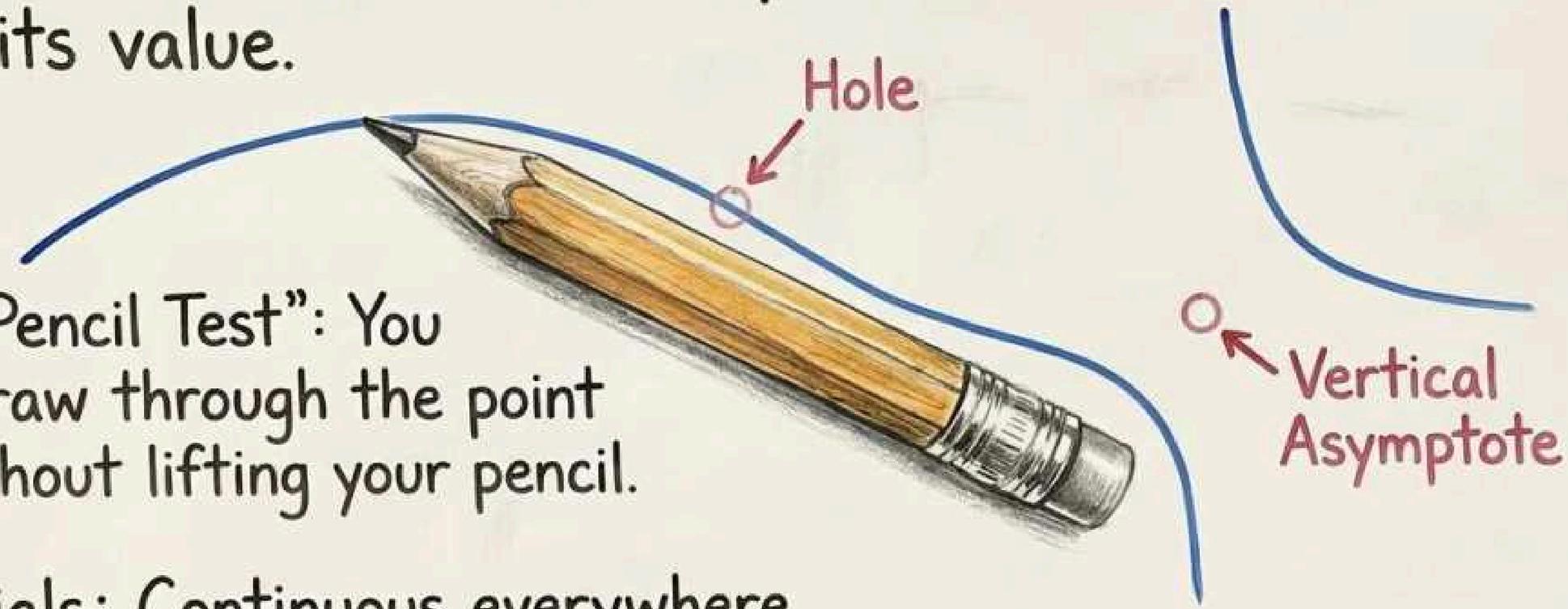
Sequence Example: $\frac{n}{n+1} \to \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\ldots$ approaches 1.

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¶ 6.15

Continuity

A function is continuous at a point if its limit there equals its value.



The "Pencil Test": You can draw through the point without lifting your pencil.

Polynomials: Continuous everywhere.

Rational Functions: Continuous except at zeros of the denominator.

(Note: The Intermediate Value Theorem requires continuity).

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Calculus Problem 1: Instantaneous Rate of Change

Average rate of change for x^2 between $x=1$ and $x=1+h$.

$$\frac{((1+h)^2-1}{h}$$

$$= \frac{1+2h+\cancel{h^2}-\cancel{1}}{h}$$

$$= \frac{\cancel{2h}+\cancel{h^2}}{\cancel{h}}$$

$$= 2+h$$

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¶ 6.17

The Derivative & The Tangent Line

The Limit (Shrinking h):

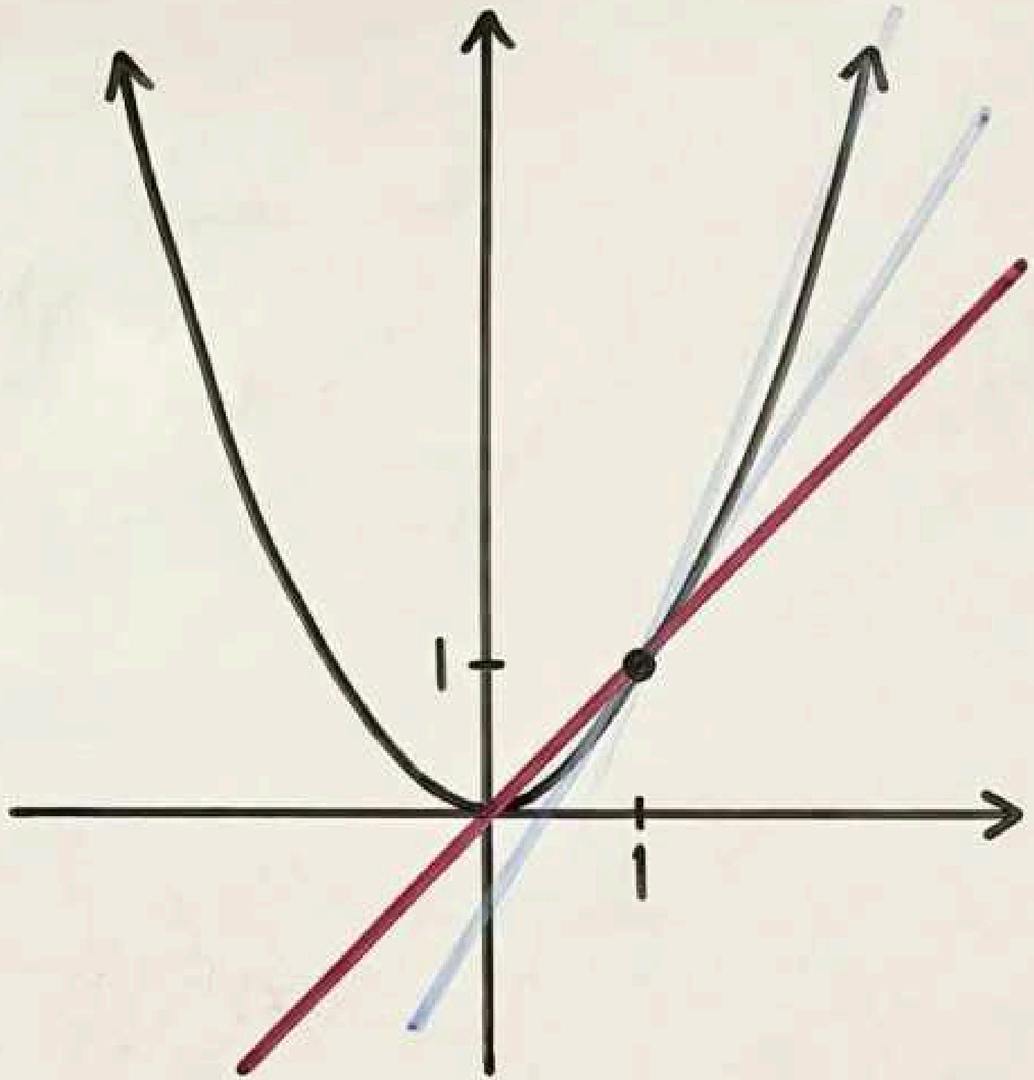
As $h \to 0$, $2 + h$ approaches 2.

Visual Meaning: This limit (2) is the slope of the tangent line at (1,1). This is the Derivative.

Line Equation: Using point (1,1) and slope m=2:

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1$$



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Physical Meaning: Falling Objects

Distance = $16t^2$ feet in t seconds.



Average Speeds (Shrinking the Interval):

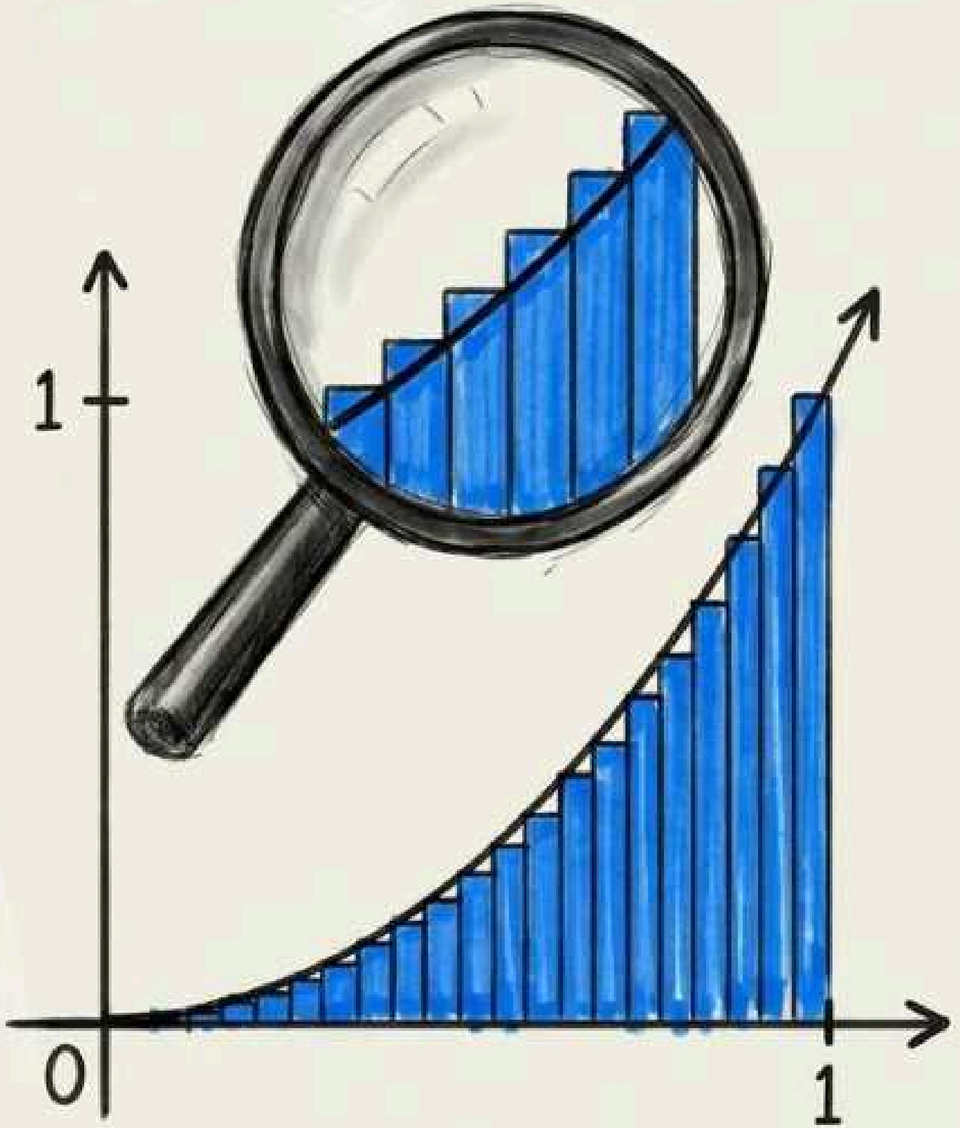
Between 1 and 2 sec: $\frac{64-16}{1} = 48$ ft/sec

Between 1 and 1.01 sec: 32.16 ft/sec

Instantaneous Speed: As the interval shrinks to zero, speed at exactly 1 second approaches 32 ft/sec.

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Calculus Problem 2: Area



The Method: Fill region under curve with n thin rectangles.

Formula: Total Area = $\frac{(n+1)(2n+1)}{6n^2}$

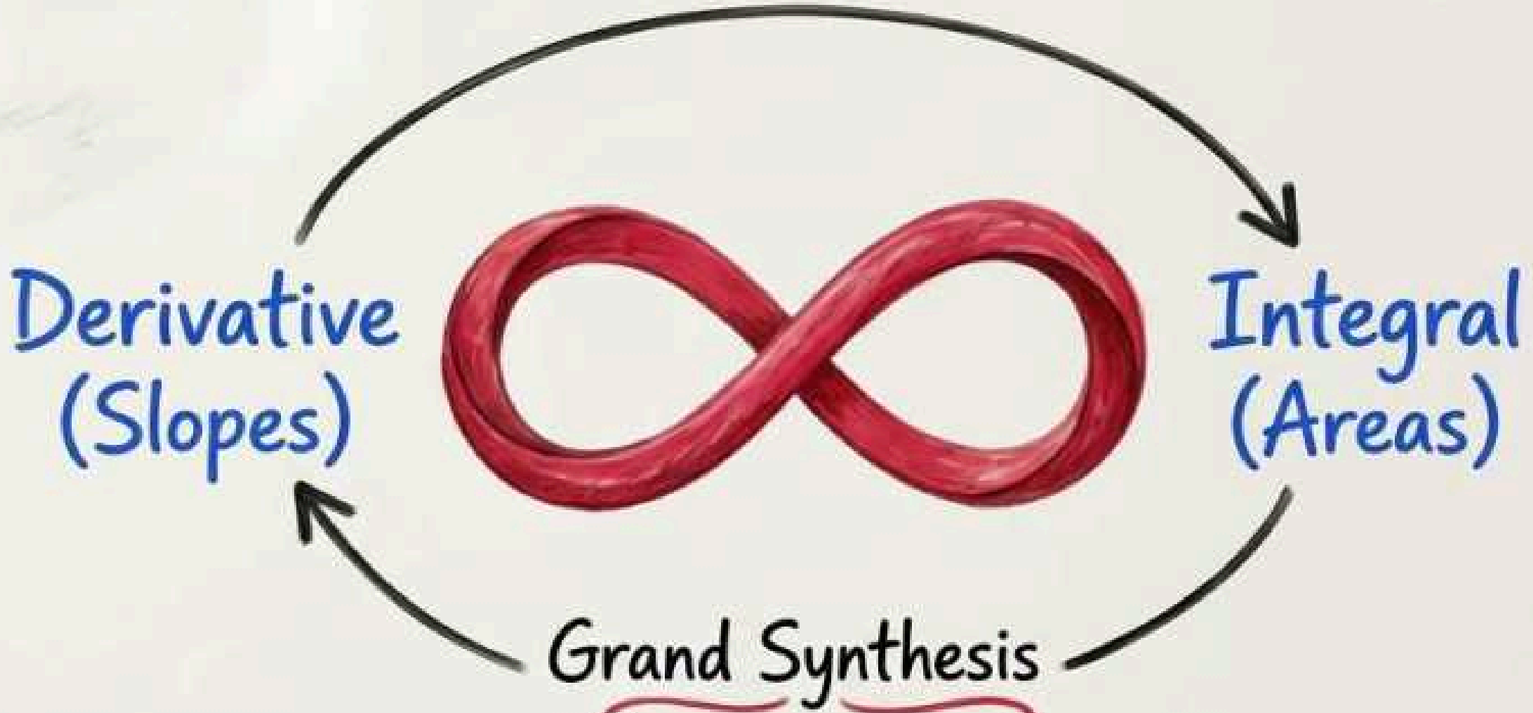
The Limit Process:

4 rectangles approx 0.469
10 rectangles approx 0.385
100 rectangles approx 0.338

Conclusion: The limit is exactly $\frac{1}{3}$.

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The Heart of Calculus



Fundamental Theorem: They undo each other (like logarithms and exponentials).

Final Challenge: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = ?$ Hint: Factor, cancel, and find out.